Analysis Prelim—July 2014—Solutions

1. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Prove that the sequence $(d(x_n, y_n))$ converges regardless of whether or not (x_n) or (y_n) converges.

   Observe that $d(x_n, y_n)$ is in the complete space $\mathbb{R}$, so all we need to show is that $(d(x_n, y_n))$ is Cauchy. Let $\varepsilon > 0$ and n_0 such that $d(x_n, x_m) + d(y_n, y_m) < \varepsilon$ whenever $n, m > n_0$. Then

   $$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_n, y_m) - d(x_m, y_m) = d(x_n, x_m) + d(y_n, y_m) < \varepsilon.$$

   As the statement is symmetric in n and m, this implies that $|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$.

2. Let $f : [0,1] \to I\!R$ satisfy

   (a) $f(0) = f(1) = 0$

   (b) $f(x) > 0, x \in (0,1)$, and

   (c) f is continuous.

   Prove that there exists $x \in (0,1)$ satisfying

$$\int_0^x f(u)\, du = x f(x).$$

   [Hint: use Intermediate Value Theorem]

Consider the continuous function

$$g(x) = x f(x) - \int_0^x f(u)\, du.$$

As continuous functions attain a maximum on a closed interval, there is an $x^* \in (0,1)$ for which $f(x)$ is maximized. Then

$$g(x^*) = x^* f(x^*) - \int_0^{x^*} f(u)\, du > x^* f(x^*) - \int_0^{x^*} f(x^*)\, du = 0,$$

where the inequality follows from the fact that by continuity there exists an $\varepsilon > 0$ such that $f(u) < f(x^*)/2$ for all $u \in [0, \varepsilon]$. Similarly, we have

$$g(1) = -\int_0^1 f(u)\, du < 0,$$

as there exists an $\varepsilon > 0$ such that $f(u) > f(x^*)/2$ for all $u \in [x^* - \varepsilon, x^*]$. By the intermediate value theorem, there exists $x \in (x^*, 1)$ with $g(x) = 0$.

3. Consider the following proposition:

   Every bounded continuous real-valued function on $I\!R$ attains its maximum.

   The following argument has an error. Find the error and provide a counterexample that the argument indeed fails at that point:

   Let $f(x) \leq M$, where M is some constant, and let $f^* = \sup\{f(x) : x \in I\!R\}$. Clearly, $f^* \leq M$. Now let $x_n \to x^*$ such that $f(x_n) \to f^*$. Then, since f is continuous, $f(x_n) \to f(x^*)$, so $f(x^*) = f^*$. Hence, x^* is where f attains its maximum.

   A sequence with $f(x_n) \to f^*$ does not necessarily have a convergent subsequence. Consider $f(x) = -\frac{1}{1+x^2}$ (clearly bounded and continuous). Then $f^* = 0$, but no subsequence of any sequence with $f(x_n) \to 0$ converges.

4. Let f and g be continuous maps of a metric space (X, d_X) into a metric space (Y, d_Y) and let E be a dense subset of X. If $g(x) = f(x)$ for all $x \in E$, prove that $g(x) = f(x)$ for all $x \in X$.

Consider the continuous function $h(x) = f(x) - g(x)$ on X. Let $x \in X$. As E is dense in X, there exists a sequence $(x_n) \in E$ with $x_n \to x$. Notice that $h(x_n) = 0$ for all n. By continuity, $h(x) = \lim h(x_n) = 0$, so $g(x) = f(x)$.

5. Prove the following theorem from Rudin:

   Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E, and there is a real number M such that

   $$\|f'(\mathbf{x})\| \le M$$

   for every $\mathbf{x} \in E$. Then

   $$|f(\mathbf{b}) - f(\mathbf{a})| \le M|\mathbf{b} - \mathbf{a}|$$

   for all $\mathbf{a} \in E$, $\mathbf{b} \in E$.

   See Rudin.

6. Let F be an equicontinuous set of functions from a metric space (X, d_X) to metric space (Y, d_Y). Let $\bar{F}$ be the set of functions defined as pointwise limits of sequences of functions in F. Show that $\bar{F}$ is equicontinuous.

Let $x \in X$, $\varepsilon > 0$, and $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $f \in F$ and $d_X(x, y) < \delta$. Let $g \in \bar{F}$, and let $(f_n) \in F$ converging pointwise to g.

Let $y \in X$ with $d_X(x, y) < \delta$. Let n be large enough such that $d_Y(f_n(x), g(x)) + d_Y(f_n(y), g(y)) < \varepsilon$. Then

$$d_Y(g(x), g(y)) < d_Y(f_n(x), g(x)) + d_Y(f_n(y), f_n(y)) + d_Y(f_n(y), g(y)) < 2\varepsilon.$$

7. Let ℓ^1 be the metric space of all real sequences, $x = (\xi_j)$, such that $\sum_{j=1}^{\infty} \xi_j$ converges absolutely and where the distance between two sequences, $x = (\xi_j)$ and $y = (\eta_j)$, is given by

$$d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|.$$

Let ℓ^∞ be the metric space of all bounded sequences and where the distance between two sequences x and y is given by $d(x, y) = \sup_j |\xi_j - \eta_j|$.

We know that ℓ^1 and ℓ^∞ are metric spaces and that $\ell^1 \subset \ell^\infty$.

Is ℓ^1 closed in ℓ^∞? If yes, prove it. If not, provide a counterexample.

No. Consider $x_n = (\xi_i^n)$, where

$$\xi_i^n = \begin{cases} \frac{1}{i}, & \text{if } i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, each x_n is in ℓ^1, and $x_n \to (\frac{1}{i})$ in ℓ^∞. But $\sum \frac{1}{i} = \infty$, so $(\frac{1}{i}) \notin \ell^1$.