University of Colorado Denver
Department of Mathematical and Statistical Sciences

Applied Analysis Preliminary Exam
July 15, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best.
- **Solve 4 of the problems 1–5 and 2 of the problems 6–8. We will only grade the first 4 problems attempted from 1–5 and the first 2 attempted from problems 6–8.**
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let $A \subset \mathbb{R}$ be a nonempty set that is bounded above, and let a be the least upper bound of A. Show that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} a_n = a$, and there is no sequence $\{b_n\}_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} b_n > a$.

2. Show that a subspace of a separable metric space is separable.

3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable real-valued functions with $|f_n'(x)| \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$. Prove that the limit function is continuous on $\mathbb{R}$.

4. (a) Prove a form of the Weierstrass M-Test for real-valued functions: Assume that $\{f_n\}$ is a sequence of real-valued functions defined on some set E, and denote $M_n = \sup_{x\in E} |f_n(x)|$ for all n. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely.

   (b) Provide an example showing that the converse is not true.

5. Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, where A is an open convex set and $f'(x) = 0$ for every $x \in A$. Prove that f is constant in A.

6. Suppose $f(x)$ is continuous on $[0, 1]$. Prove that $\lim_{M\to\infty} \int_0^1 Me^{-Mx} f(x)dx = f(0)$. You can use all the standard properties of the exponential function from Calculus without proving them here.

7. In this problem, you may use part (a) to help prove part (b) and use both (a) and (b) to help prove part (c) even if you cannot prove these parts (but you will only receive partial credit for such responses).

   (a) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers. For $n \geq 0$, let $A_n = \sum_{k=0}^{n} a_k$ denote the n-th partial sum of $\sum_{k=0}^{\infty} a_k$, and let $A_{-1} = 0$. Prove that if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

   (b) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, the partial sums A_n of $\sum a_n$ form a bounded sequence, and $\{b_n\}$ is a positive sequence that monotonically converges to zero. Prove that $\sum a_n b_n$ converges.

   (c) Show that the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^a}$ converges for any $a > 0$ and all $x \in \mathbb{R}$. *Hint: Euler's identity gives* $e^{i\theta} = \cos\theta + i \sin\theta$.

8. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$. Let $G = \{(x, f(x)) : x \in [a, b]\} \subset \mathbb{R}^2$ denote the graph of f. Show that f is continuous if and only if G is compact in $\mathbb{R}^2$.