Applied Analysis prelim July 15, 2016, with solutions

Solve 4 of the problems 1-5 and 2 of the problems 6-8. We will only grade the first 4 problems attempted from1-5 and the first 2 attempted from problems 6-8.

1. Let $A \subset \mathbb{R}$ be a nonempty set that is bounded from above, and let a be the least upper bound of A. Show that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} a_n = a$, and there is no sequence $\{b_n\}_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} b_n > a$.

   Solution.

   Since A is bounded from above, $a = \sup A \in \mathbb{R}$ by the completeness axiom.

   For each $n \in \mathbb{N}$, $a - 1/n < a$ cannot be an upper bound for A and there exists $a_n \in A$ such that $a - 1/n < a_n \leq a$.

   Since $\{a-1/n\}$ and $\{a\}$ both converge to a as $n \to \infty$, $\{a_n\}$ also converges to a by the squeeze theorem.

   Let $\{b_n\} \subset A$ such that $\lim_{n\to\infty} b_n = b$ exists.

   Since $b_n \in A$, we have that $b_n \leq a$ for each n. Therefore, $\lim_{n\to\infty} b_n \leq a$.

2. Show that a subspace of a separable metric space is separable.

   Solution.

   Let (X, d) denote a separable metric space and $Y \subset X$ be a subspace.

   Since X is separable, there exists a countable dense subset of X that we denote by A. Below, we construct a countable $C \subset Y$.

   Since $\mathbb{Q} \cap (0, 1)$ is countable, and the Cartesian product of countable sets is countable, it follows that

   $$\{B_q(a) \subset X \,:\, a \in A, q \in \mathbb{Q} \cap (0,1)\}$$

   is countable. For each (a, q) such that $B_q(a) \cap Y \neq \emptyset$, choose $c \in B_q(a) \cap Y$, and denote by C the set of all such points. The set $C \subset Y$ and is countable by construction.

   We now show that C is dense in Y.

   Let $y \in Y$ and $\varepsilon > 0$. There exists $q \in \mathbb{Q} \cap (0, 1)$ such that $q < \varepsilon/2$. Since $y \in Y \subset X$ and A is dense in X, there exists $a \in A$ such that $d(y, a) < q$. Thus, $y \in B_q(a) \cap Y$, so there exists $c \in C$ chosen from $B_q(a) \cap Y$. By the triangle inequality,

   $$d(y, c) \leq d(y, a) + d(c, a) < q + q < \varepsilon. \ \Box$$

3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable real-valued functions with $|f_n'(x)| \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$. Prove that the limit function is continuous on $\mathbb{R}$.

   Solution. Denote by f the limit function. We in fact prove that f is uniformly continuous. For any $x \neq y$, we have by the mean value theorem that $f_n(x) - f_n(y) = f_n'(\zeta)(x - y)$ for some ζ between x and y. Since $|f_n'(\zeta)| \leq 1$ for all n, it follows that for each n,

   $$|f_n(x) - f_n(y)| \leq |x - y| \ \ \forall x, y \in \mathbb{R}.$$

   Let $\varepsilon > 0$.

   Choose $\delta = \varepsilon/3$.

   Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. Then,

   $$|f_n(x) - f_n(y)| < \varepsilon/3$$

   for all n. Since $\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} f_n(y)$, there exists N_1 and N_2 such that for $n \geq N_1$,

   $$|f_n(x) - f(x)| < \varepsilon/3$$

and for $n \geq N_2$,

$$|f_n(y) - f(y)| < \varepsilon/3,$$

respectively.

Choose $n = \max\{N_1, N_2\}$. Then,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \ \square$$

Another solution.

Denote by f the pointwise limit of $\{f_n\}$, and x be a real number. Consider a closed interval $[a, b]$ such that x is in its interior (for example $[x-1, x+1]$). By the mean value theorem, the functions $\{f_n\}$ are uniformly equicontinuous on the interval $[a, b]$, and by the Arzèla-Ascoli theorem, there exists a subsequence $\{f_{n_k}\}$ which converges uniformly on $[a, b]$. Then $\{f_{n_k}\}$ converges pointwise to the same limit on $[a, b]$, so from the uniqueness of limit, $f_{n_k} \rightrightarrows f$ on $[a, b]$. Since the uniform limit of continuous functions is continuous, f is continuous on $[a, b]$ (with respect to $[a, b]$ as a metric space). Since x is an interior point of $[a, b]$ relative to $\mathbb{R}$, f is continous at x (relative to $\mathbb{R}$). $\square$

Note:

It is not generally true that there exists a subsequence $\{f_{n_k}\}$ convergent uniformly on the whole $\mathbb{R}$.

4. (a) Prove a form of the Weierstrass M-Test for real-valued functions: Assume that $\{f_n\}$ is a sequence of real-valued functions defined on some set E, and $M_n = \sup\{|f_n(x)| | x \in E\}$ exists for all n. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely.

   (b) Provide an example showing that the converse is not true.

   Solution.

   (a) Let $\epsilon > 0$.

   Since $\sum_{n=1}^{\infty} M_n$ converges, there exists N such that for all $m \geq n \geq N$,

$$\sum_{k=n}^{m} M_k \leq \epsilon.$$

   It follows that

$$\left| \sum_{k=n}^{m} f_k(x) \right| \leq \sum_{k=n}^{m} M_k \leq \epsilon.$$

   Thus, the sequence of partial sums for $\sum_{n=1}^{\infty} f_n$ is a uniformly Cauchy sequence. A standard theorem then implies this sequence converges uniformly. $\square$

   (b) Let $f_n = (1/n)\,\chi_{[n-1,n)}$ for each $n \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[0, \infty)$ since for $N \in \mathbb{N}$, $\sup_{x \in [0,\infty)} |\sum_{n=1}^{\infty} f_n - \sum_{n=1}^{N} f_n(x)| = 1/(N+1) \to 0$ as $N \to \infty$. However, $M_n = 1/n$ so $\sum_{n=1}^{\infty} M_n$ fails to converge. $\square$

5. Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, where A is an open convex set and $f'(x) = 0$ for every $x \in A$. Prove that f is constant in A.

   Solution.

   Let $x, y \in A$.

   Since A is convex, the line segment connecting x to y lies in A.

   Since A is convex, it is also connected, so the mean value theorem in several variables implies there exists a z on the line segment connecting x to y such that

$$f(y) - f(x) = f'(z) \cdot (y - z).$$

   Since $f'(z) = 0$ for all $z \in A$, we have that $f(x) = f(y)$. Thus, f is constant on A. $\square$

6. Suppose $f(x)$ is continuous on $[0,1]$. Prove that $\lim_{M\to\infty}\int_0^1 Me^{-Mx}f(x)dx = f(0)$. You can use all the standard properties of the exponential function from Calculus without proving them here.

Solution.

The idea is that for large M, the integral is concentrated near $x \approx 0$. We need to express this idea rigorously. Because $f(x)$ is continuous on $[0,1]$ it is bounded on $[0,1]$: there exists $C > 0$ such that $|f(x)| < C$. For every $\varepsilon > 0$ there exists $1 > \delta > 0$ such that for all $0 \le x < \delta$, $|f(x) - f(0)| < \varepsilon$.

$$\int_0^1 Me^{-Mx}f(x)dx = \int_0^\delta Me^{-Mx}f(x)dx + \int_\delta^1 Me^{-Mx}f(x)dx \tag{1}$$

For the first integral on the last line

$$\begin{aligned}\int_0^\delta Me^{-Mx}f(x)dx &= \int_0^\delta Me^{-Mx}f(0)dx + \int_0^\delta Me^{-Mx}(f(x)-f(0))dx \\ &= f(0)(1-e^{-M\delta}) + \int_0^\delta Me^{-Mx}(f(x)-f(0))dx\end{aligned}$$

Now

$$\left|\int_0^\delta Me^{-Mx}(f(x)-f(0))dx\right| \le \int_0^\delta Me^{-Mx}|f(x)-f(0)|dx \le \varepsilon(1-e^{-M\delta}) < \varepsilon$$

For the second integral in (1)

$$\left|\int_\delta^1 Me^{-Mx}f(x)dx\right| \le \int_\delta^1 Me^{-Mx}|f(x)|dx \le C(e^{-M\delta}-e^{-M}) < Ce^{-M\delta}$$

Combining it all now

$$\left|\int_0^1 Me^{-Mx}f(x)dx - f(0)\right| < |f(0)|e^{-M\delta} + Ce^{-M\delta} + \varepsilon \le 2Ce^{-M\delta} + \varepsilon$$

For every $\varepsilon > 0$ choose δ as before and $N = \frac{1}{\delta}\ln(2C/\varepsilon)$. Because $e^{-M\delta}$ is decreasing function of M, for any $M \ge N$ $e^{-M\delta} \le e^{-N\delta} = \varepsilon$ and

$$\left|\int_0^1 Me^{-Mx}f(x)dx - f(0)\right| < 2\varepsilon \ \square$$

Another Solution.

By standard results we have that $Me^{-Mx}f(x)$ is integrable on $[0,1]$. We observe that

$$\begin{aligned}\int_0^1 Me^{-Mx}f(x)\,dx &= \int_0^1 Me^{-Mx}(f(x)-f(0)+f(0))\,dx \\ &= \int_0^1 Me^{-Mx}f(0)\,dx + \int_0^1 Me^{-Mx}(f(x)-f(0))\,dx.\end{aligned}$$

Using the last line above, we show that the limit as $M\to\infty$ of the first integral is $f(0)$ and the limit as $M\to\infty$ of the second integral is zero.

Clearly, by standard Calculus results, the first integral becomes

$$\int_0^1 Me^{-Mx}f(0)\,dx = f(0)[1-e^{-M}] \to f(0) \text{ as } M\to\infty.$$

Because $f(x)$ is continuous on $[0,1]$ it is bounded on $[0,1]$ by a standard result, so there exists $C > 0$ such that $|f(x)| < C$. Consider any $\varepsilon > 0$. Since f is continuous, there exists $\delta \in (0,1)$ such that for all $0 \le x < \delta$, $|f(x) - f(0)| < \varepsilon/2$. For all $x > 0$, $e^{-Mx} \to 0^+$ as $M \to \infty$, so there exists $N > 0$ such that for all $M \ge N$, $e^{-M\delta} < \varepsilon/(8C)$. By the triangle inequality, it follows that

$$\left| \int_0^1 Me^{-Mx}(f(x) - f(0))\, dx \right| \le \left| \int_0^\delta Me^{-Mx}(f(x) - f(0))\, dx \right| + \left| \int_\delta^1 Me^{-Mx}(f(x) - f(0))\, dx \right|.$$

Since $|f(x) - f(0)| < \varepsilon/2$ for $x \in (0, \delta)$ and $0 < 1 - e^{-M\delta} < 1$, we use a standard result from integration to get

$$\begin{aligned} \left| \int_0^\delta Me^{-Mx}(f(x) - f(0))\, dx \right| &\le \int_0^\delta \left| Me^{-Mx}(f(x) - f(0)) \right| \, dx \\ &\le \varepsilon/2[1 - e^{-M\delta}] \\ &< \varepsilon/2. \end{aligned}$$

Since $|f(x) - f(0)| < 2C$ for all $x \in [0,1]$ and $0 < e^{-M\delta} - e^{-M} < 2e^{-M\delta}$, we again use a standard result from integration to get

$$\begin{aligned} \left| \int_\delta^1 Me^{-Mx}(f(x) - f(0))\, dx \right| &\le \int_\delta^1 \left| Me^{-Mx}(f(x) - f(0)) \right| \, dx \\ &\le 2C[e^{-M\delta} - e^{-M}] \\ &< \varepsilon/2. \end{aligned}$$

It follows that for all $M \ge N$, we have

$$\left| \int_0^1 Me^{-Mx}(f(x) - f(0))\, dx \right| < \varepsilon,$$

and the conclusion follows. □

7. In this problem, you may use part (a) to help prove part (b) and use both (a) and (b) to help prove part (c) even if you cannot prove these parts (but you will only receive partial credit for such responses).

   (a) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers. For $n \ge 0$, let $A_n = \sum_{k=0}^n a_k$ denote the n-th partial sum of $\sum_{k=0}^\infty a_k$, and let $A_{-1} = 0$. Prove that if $0 \le p \le q$, we have

   $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

   (b) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, the partial sums A_n of $\sum a_n$ form a bounded sequence, and $\{b_n\}$ is a positive sequence that monotonically converges to zero. Prove that $\sum a_n b_n$ converges.

   (c) Show that the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^a}$ converges for any $a > 0$ and all $x \in \mathbb{R}$. *Hint: Euler's identity gives* $e^{i\theta} = \cos\theta + i \sin\theta$.

Solution.

(a) We have $a_n = A_n - A_{n-1}$, thus

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1})\, b_n \\ &= (A_p - A_{p-1})\, b_p + (A_{p+1} - A_p)\, b_{p+1} + \cdots + (A_{q-1} - A_{q-2})\, b_{q-1} + (A_q - A_{q-1})\, b_q \\ &= -A_{p-1} b_p + A_p\,(b_p - b_{p+1}) + \cdots + A_{q-1}\,(b_{q-1} - b_q) + A_q b_q \\ &= -A_{p-1} b_p + \sum_{n=p}^{q-1} A_p\,(b_n - b_{n+1}) + A_q b_q\end{aligned}$$

by rearranging the terms ("telescoping"). □

(b) We will show that the partial sums of $\sum a_n b_n$ are Cauchy. Let $q > p > 0$ be integers. By part (a),

$$\begin{aligned}\sum_{n=1}^{q} a_n b_n - \sum_{n=1}^{p} a_n b_n &= \sum_{n=p+1}^{q} a_n b_n \\ &= -A_p b_{p+1} + \sum_{n=p+1}^{q-1} A_p\,(b_n - b_{n+1}) + A_q b_q.\end{aligned}$$

Since the sequence $\{A_n\}$ is bounded, there exists C such that $|A_n| \le C$ for all n. By the triangle inequality,

$$\begin{aligned}\left|\sum_{n=1}^{q} a_n b_n - \sum_{n=1}^{p} a_n b_n\right| &\le |A_p|\,|b_{p+1}| + \sum_{n=p+1}^{q-1} |A_p|\,|b_n - b_{n+1}| + |A_q|\,|b_q| \\ &\le C b_{p+1} + C \sum_{n=p+1}^{q-1} (b_n - b_{n+1}) + C b_q \\ &= C b_{p+1} + C b_{p+1} - C b_q + C b_q \\ &= 2C b_{p+1}.\end{aligned}$$

where we used the fact that $b_n \ge b_{n-1} \ge 0$, thus $|b_{p+1}| = b_{p+1}$, $|b_n - b_{n+1}| = b_n - b_{n+1}$, and $|b_q| = b_q$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} b_n = 0$, there exists N such that for all $n > N$, $b_n < \frac{\varepsilon}{2C}$. Then, for any $q > p > N$,

$$\left|\sum_{n=1}^{q} a_n b_n - \sum_{n=1}^{p} a_n b_n\right| < 2C b_{p+1} < 2C \frac{\varepsilon}{2C} = \varepsilon. \ \square$$

(c) Let $x \in \mathbb{R}$, $\dot{a} > 0$, and set in part (b), $a_n = \sin nx$ and $b_n = \frac{1}{n^a}$. Clearly, $b_n \searrow 0$. To estimate the partial sums of $\sum_{a_n}$, first note that if $x = 2k\pi$ with integer k, then $\sin nx = \sin 2nk\pi = 0$, all $a_n = 0$, and zero series converges. Let $x \ne 2k\pi$ for any integer k. By Euler's formula,

$$\sum_{n=1}^{N} a_n = \operatorname{Im} \sum_{n=1}^{N} e^{inx}.$$

By the formula for the sum of geometric sequence and properties of complex numbers to get

$$\sum_{n=1}^{N} e^{inx} = e^{ix} \sum_{n=0}^{N-1} e^{inx} = e^{ix} \frac{1 - e^{iNx}}{1 - e^{ix}}.$$

and

$$\left|\sum_{n=1}^{N} a_n\right| = \left|\mathrm{Im}\sum_{n=1}^{N} e^{inx}\right| \leq \left|\sum_{n=1}^{N} e^{inx}\right|$$
$$= \left|e^{ix}\right| \frac{1+\left|e^{iNx}\right|}{\left|1-e^{ix}\right|} \leq 1\frac{1+1}{\left|1-e^{ix}\right|},$$

where $e^{ix} \neq 1$ because $x \neq 2k\pi$. □

8. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$. Let $G = \{(x, f(x)) : x \in [a, b]\} \subset \mathbb{R}^2$ denote the graph of f. Show that f is continuous if and only if G is compact in $\mathbb{R}^2$.

   Solution using sequential compactness in both directions.

   Suppose f is continuous.

   Consider any sequence of points $\{(x_n, y_n)\} \subset G$.

   By definition of G, $y_n = f(x_n)$ for each n and $\{x_n\} \subset [a, b]$.

   Since $[a, b]$ is closed and bounded in $\mathbb{R}$, there exists a subsequence $\{x_{n_k}\}$ that converges to a point $c \in [a, b]$. Choose such a subsequence.

   By the continuity of f, $y_{n_k} = f(x_{n_k}) \to f(c)$, and again by the definition of G, we have that $(c, f(c)) \in G$, so G is compact. □

   Now suppose G is compact.

   Consider any sequence $\{x_n\} \in [a, b]$ that converges to $x \in [a, b]$, then $\{(x_n, f(x_n))\} \subset G$.

   Since G is compact, there exists a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ that converges to $(y, f(y)) \in G$.

   By properties of the Euclidean metric,

$$|x_{n_k} - y| \leq \|(x_{n_k}, f(x_{n_k})) - (y, f(y))\| \to 0,$$

   so $x_{n_k} \to y$, but $x_{n_k} \to x$, and by the uniqueness of limits, $y = x$. Again, by properties of the Euclidean metric,

$$|f(x_{n_k}) - f(y)| \leq \|(x_{n_k}, f(x_{n_k})) - (y, f(y))\| \to 0.$$

   Thus, $f(x_{n_k}) \to f(x) = f(y)$.

   We now show that the whole sequence $f(x_n) \to f(x)$ by contradiction.

   Assume $f(x_n) \not\to f(x)$.

   Then, there exists an $\epsilon > 0$ and a subsequence $\{f(x_{n_k})\}$ such that $|f(x_{n_k}) - f(x)| \geq \epsilon$, which implies there are no subsequences of $\{f(x_{n_k})\}$ that converge to $f(x)$. Choose such a subsequence and the corresponding subsequence $\{x_{n_k}\}$.

   Since $x_{n_k} \to x \in [a, b]$, the exact same argument above shows that $f(x_{n_{k_l}}) \to f(x)$, a contradiction. □

   Another solution to the forward direction:

   G is the image of $[a, b]$ under the continuous mapping $x \mapsto (x, f(x))$, and the continuous image of a compact set is compact by a standard theorem.