Analysis Prelim—January 2015

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your relevant work. Rewrite your solutions, if necessary, so they are neat and easy to read.
- **Only write on one side of each sheet.**
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Problems

1. Let (X, d) be a complete metric space. Prove that a subspace of (X, d) is complete if and only if it is closed.

2. Let X be any nonempty set and d_X the discrete metric on X. Let (Y, d_Y) be a metric space. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ denote a family of functions $f_\alpha : X \to Y$ for each $\alpha \in \mathcal{A}$. Prove that $\mathcal{F}$ is uniformly equicontinuous.

3. Let $x = (\xi_i),\ y = (\eta_j)$ denote points in ℓ^2.

   (a) Prove that the function $d_1(x, y)$ defined by
   $$d_1(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|$$
   is *not* a metric on ℓ^2.

   (b) Prove that the function $d_2(x, y)$ defined by
   $$d_2(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 \right)^{1/2}$$
   is a metric on ℓ^2. Hint: Use Minkowski's inequality.

4. Consider the metric space (ℓ^2, d_2) from problem 3 where $x = (\xi_i) \in \ell^2$. For each $n \in \mathbb{N}$, let $f_n : \ell^2 \to \mathbb{R}$ be defined by
   $$f_n(x) = f_n((\xi_i)) = \sum_{i=n}^{\infty} |\xi_i|^2$$
   Prove that (a) $f_n(x) \to 0$ for every $x \in \ell^2$, but (b) the convergence to the function $f(x) = 0$ is *not* uniform.

5. Let (X, d_X) and (Y, d_Y) be metric spaces. For each $n \in \mathbb{N}$, suppose $f_n : X \to Y$ is a continuous function. Prove if Y is complete and (f_n) is uniformly Cauchy, then there exists continuous $f : X \to Y$ such that $f_n \to f$ uniformly.

6. Let (g_n) and (f_n) denote sequences of Riemann integrable functions, $[a, b] \to \mathbb{R}$. Suppose

   (i) $\forall m \in \mathbb{N}$, $\lim_{n\to\infty} \int_a^b |f_m g_n| dx = 0$,

   (ii) $f_n \to f$ uniformly on $[a, b]$, and

   (iii) the sequence $(\int_a^b |g_n| dx)$ is bounded.

   Prove that
   $$\lim_{n\to\infty} \int_a^b f g_n dx = 0.$$

7. Prove the following standard theorem. A compact metric space is separable.

Useful definitions and theorems

1. A family of functions $\{f_\alpha\}$, $\alpha \in \mathcal{A}$ maping metric spaces $(X, d_X) \to (Y, d_Y)$ is equicontinuous if $\forall x \in X$,and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_Y(f_\alpha(x), f_\alpha(y)) < \epsilon$ for every $\alpha \in \mathcal{A}$ and every $y \in X$ with $d_X(x, y) < \delta$.

2. The discrete metric d on X assigns $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$.

3. For $1 \leq p < \infty$, the space ℓ^p is the set of sequences $(x_i) = (x_1, x_2, \ldots) \subset \mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$.

4. Minkowski's inequality: Let $x = (\xi_j) \in \ell^p$ and $y = (\eta_j) \in \ell^p$. Then

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p \right)^{1/p}.$$

5. From Rudin, Theorem 7.16: Suppose f_n is Riemann integrable on $[a, b]$ for $n = 1, 2, \ldots$ and $f_n \to f$ uniformly on $[a, b]$, then f is Riemann integrable on $[a, b]$ and $\int_a^b f_n dx \to \int_a^b f dx$.