Analysis Prelim—January 2015

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your work.
- **Only write on one side of each sheet.**
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Problems

1. Let (X, d) be a complete metric space. Prove that a subspace of (X, d) is complete if and only if it is closed.

   Proof: Let $A \subset X$ be any metric subspace. Suppose A is complete. If $x \in X$ is a limit point of A, then there exists a sequence $(x_n) \subset A$ such that $x_n \to x$ in X. Since limit points are unique, convergent sequences are Cauchy, and A is complete, this implies $x \in A$. Now suppose A is closed and consider an arbitrary Cauchy sequence $(x_n) \subset A$. Since X is complete, there exists $x \in X$ such that $x_n \to x$. If the sequence does not eventually become constant (which implies $x_n = x$ for all sufficiently large n and $x \in A$), then x is a limit point of A. Since A is closed, x belongs to A. □

2. Let X be any nonempty set and d_X the discrete metric on X. Let (Y, d_Y) be a metric space. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ denote a family of functions $f_\alpha : X \to Y$ for each $\alpha \in \mathcal{A}$. Prove that $\mathcal{F}$ is uniformly equicontinuous.

   Proof: Consider any $\epsilon > 0$ and fix any $\delta \in (0, 1)$. For $x, y \in X$ with $d_X(x, y) < \delta$, $x = y$ since d_X is the discrete metric. Thus, for any $\alpha \in \mathcal{A}$, and $x, y \in X$ with $d_X(x, y) < \delta$, $d_Y(f_\alpha(x), f_\alpha(y)) = d_Y(f_\alpha(x), f_\alpha(x)) = 0 < \epsilon$, which proves the result. $\square$

3. Let $x = (\xi_i),\ y = (\eta_j)$ denote points in ℓ^2.

(a) Prove that the function $d_1(x, y)$ defined by

$$d_1(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|$$

is *not* a metric on ℓ^2.

(b) Prove that the function $d_2(x, y)$ defined by

$$d_2(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 \right)^{1/2}$$

is a metric on ℓ^2. Hint: Use Minkowski's inequality.

Proof (a): Note that $x = (1/j) \in \ell^2$ and $y = (0) \in \ell^2$ (i.e., y is the sequence of all 0's), yet $d_1(x, y) = \sum_{j=1}^{\infty} 1/j = +\infty$, so d_1 does not map ℓ^2 into $[0, \infty)$ and is not a metric. □

Proof (b): Consider any $x = (\xi_i), y = (\eta_j) \in \ell^2$. Clearly, $d_2(x, y) \geq 0$ by properties of the absolute value. Applying Minkowski's inequality to x and $-y$, we have

$$(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2} \leq (\sum_{k=1}^{\infty} |\xi_k|^2)^{1/2} + (\sum_{m=1}^{\infty} |\eta_m|^2)^{1/2} < \infty$$

Thus, $d_2(x, y) \in [0, \infty)$.

Suppose $d_2(x, y) = 0 \Leftrightarrow d_2^2(x, y) = 0 \Leftrightarrow \sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 = 0 \Leftrightarrow |\xi_j - \eta_j|^2 = 0$ for all $j \in \mathbb{N}$ $\Leftrightarrow |\xi_j - \eta_j| = 0$ for all $j \in \mathbb{N} \Leftrightarrow \xi_j = \eta_j$ for all $j \Leftrightarrow x = y$.

For any $a, b \in \mathbb{R}$, $|a - b| = |b - a|$. It follows that $d_2(x, y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2} = (\sum_{j=1}^{\infty} |\eta_j - \xi_j|^2)^{1/2} = d(y, x)$.

Now consider any $z = (\zeta_k) \in \ell^2$. The triangle inequality of the absolute value on $\mathbb{R}$ implies that for each $j \in \mathbb{N}$ we have $|\xi_j - \eta_j|^2 \leq [|\xi_j - \zeta_j| + |\zeta_j - \eta_j|]^2$. Note that Minkowski's inequality implies that the term-wise addition or subtraction of any two sequences in ℓ^2 is also in ℓ^2. Thus, we apply Minkowski's inequality below to get

$$\begin{aligned} d_2(x, y) &= (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2} \\ &\leq (\sum_{j=1}^{\infty} [|\xi_j - \zeta_j| + |\zeta_j - \eta_j|]^2)^{1/2} \\ &\leq (\sum_{k=1}^{\infty} |\xi_k - \zeta_k|^2)^{1/2} + (\sum_{m=1}^{\infty} |\zeta_m - \eta_m|^2)^{1/2} \\ &= d(x, z) + d(z, y) \ \square \end{aligned}$$

4. Consider the metric space (ℓ^2, d_2) from problem 3 where $x = (\xi_i) \in \ell^2$. For each $n \in \mathbb{N}$, let $f_n : \ell^2 \to \mathbb{R}$ be defined by

$$f_n(x) = f_n((\xi_i)) = \sum_{i=n}^{\infty} |\xi_i|^2$$

Prove that (a) $f_n(x) \to 0$ for every $x \in \ell^2$, but (b) the convergence to the function $f(x) = 0$ is *not* uniform.

Proof (a): Consider any $x = (\xi_i) \in \ell^2$ and $\epsilon > 0$. Since $x \in \ell^2$, $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$, which implies that there exists N such that $\sum_{i=n}^{\infty} |\xi_i|^2 < \epsilon$ for $n > N$. Thus, for $n > N$,

$$|f_n(x) - 0| = \sum_{i=n}^{\infty} |\xi_i|^2 < \epsilon \ \square$$

Proof (b): Let $\epsilon = 1/2$ and consider any $m \in \mathbb{N}$. Consider the sequence $(x_n) = ((\xi_{i,n})) \subset \ell^2$ defined by $\xi_{i,n} = 1$ if $i = n$ and $\xi_{i,n} = 0$ if $i \neq n$. Then, $f_m(x_n) = 1$ for any $n > m$, so $|f_m(x_n) - f(x_n)| = 1 > \epsilon$ for any $n > m$ and the convergence cannot be uniform. $\square$

5. Let (X, d_X) and (Y, d_Y) be metric spaces. For each $n \in \mathbb{N}$, suppose $f_n : X \to Y$ is a continuous function. Prove if Y is complete and (f_n) is uniformly Cauchy, then there exists continuous $f : X \to Y$ such that $f_n \to f$ uniformly.

   Proof: Consider any $x \in X$, then $(f_n(x)) \subset Y$ is Cauchy since (f_n) is uniformly Cauchy. Since Y is complete, there exists $y \in Y$ such that $f_n(x) \to y$. Define $f : X \to Y$ by this point-wise limit, i.e. $f(x) = \lim_{n\to\infty} f_n(x) = y$.

   We now prove that the convergence is uniform. Let $\epsilon > 0$ be given. Since (f_n) is uniformly Cauchy, there exists N such that for any $n, m > N$, $d_Y(f_n(x), f_m(x)) < \epsilon/2$ for every $x \in X$. Thus, for fixed $n > N$, since $f_m(x) \to f(x)$, by the (sequential) continuity of the metric $d_Y(f_n(x), f(x)) \le \epsilon/2 < \epsilon$. This proves the convergence is uniform.

   We now prove that the function f is continuous. Since $f_n \to f$ uniformly, there exists N such that for every $x \in X$, $d_Y(f_n(x), f(x)) < \epsilon/3$ for $n > N$. Choose such an N and let $n = N + 1$. Consider any $x \in X$. Since f_n is continuous at x, there exists $\delta > 0$ such that if $y \in X$ with $d_X(x, y) < \delta$, then $d_Y(f_n(x), f_n(y)) < \epsilon/3$. Choose such a δ. Then, for any $y \in X$ with $d_X(x, y) < \delta$,

$$d_Y(f(x), f(y)) \le d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) < \epsilon \; \square.$$

6. Let (g_n) and (f_n) denote sequences of Riemann integrable functions, $[a,b] \to \mathbb{R}$. Suppose

(i) $\forall m \in \mathbb{N}$, $\lim_{n\to\infty} \int_a^b |f_m g_n| dx = 0$,

(ii) $f_n \to f$ uniformly on $[a,b]$, and

(iii) the sequence $\{\int_a^b |g_n| dx\}$ is bounded.

Prove that

$$\lim_{n\to\infty} \int_a^b f g_n dx = 0.$$

Proof: Choose $\epsilon > 0$. Let K be a bound on $\{\int_a^b |g_n| dx\}$, and choose m large enough so that $|f_m(x) - f(x)| < \epsilon/2K(b-a)$ for $x \in [a,b]$. Then

$$\int_a^b |f g_n| dx = \int_a^b |f_m g_n - f_m g_n + f g_n| dx \leq \int_a^b |f_m g_n| dx + \int_a^b |f_m - f| g_n dx$$

By (i) we can choose N so that $n > N$ implies $\int_a^b |f_m g_n| dx < \epsilon/2$. By our choice of m and K, $\int_a^b |f_m - f||g_n| dx < \epsilon/2$. Thus, for $n > N$, $\int_a^b |f g_n| dx < \epsilon$. Since $\epsilon > 0$ was arbitrary, the result is proved.