Analysis Prelim—July 2014

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your work.
- **Only write on one side of each sheet.**
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Problems

1. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Prove that the sequence $(d(x_n, y_n))$ converges regardless of whether or not (x_n) or (y_n) converges.

2. Let $f : [0, 1] \to I\!R$ satisfy

   (a) $f(0) = f(1) = 0$
   (b) $f(x) > 0, x \in (0, 1)$, and
   (c) f is continuous.

   Prove that there exists $x \in (0, 1)$ satisfying
   $$\int_0^x f(u)\, du = x f(x).$$
   [Hint: use Intermediate Value Theorem]

3. Consider the following proposition:
   Every bounded continuous real-valued function on $I\!R$ attains its maximum.

   The following argument has an error. Find the error and provide a counterexample that the argument indeed fails at that point:
   Let $f(x) \leq M$, where M is some constant, and let $f^* = \sup\{f(x) : x \in I\!R\}$. Clearly, $f^* \leq M$. Now let $x_n \to x^*$ such that $f(x_n) \to f^*$. Then, since f is continuous, $f(x_n) \to f(x^*)$, so $f(x^*) = f^*$. Hence, x^* is where f attains its maximum.

4. Let f and g be continuous maps of a metric space (X, d_X) into a metric space (Y, d_Y) and let E be a dense subset of X. If $g(x) = f(x)$ for all $x \in E$, prove that $g(x) = f(x)$ for all $x \in X$.

5. Prove the following theorem from Rudin:

   Suppose f maps a convex open set $E \subset I\!R^n$ into $I\!R^m$, f is differentiable in E, and there is a real number M such that
   $$\|f'(\mathbf{x})\| \leq M$$
   for every $\mathbf{x} \in E$. Then
   $$|f(\mathbf{b}) - f(\mathbf{a})| \leq M|\mathbf{b} - \mathbf{a}|$$
   for all $\mathbf{a} \in E$, $\mathbf{b} \in E$.

6. Let F be an equicontinuous set of functions from a metric space (X, d_X) to metric space (Y, d_Y). Let $\bar{F}$ be the set of functions defined as pointwise limits of sequences of functions in F. Show that $\bar{F}$ is equicontinuous.

7. Let ℓ^1 be the metric space of all real sequences, $x = (\xi_j)$, such that $\sum_{j=1}^{\infty} \xi_j$ converges absolutely and where the distance between two sequences, $x = (\xi_j)$ and $y = (\eta_j)$, is given by
   $$d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|.$$
   Let ℓ^∞ be the metric space of all bounded sequences and where the distance between two sequences x and y is given by $d(x, y) = \sup_j |\xi_j - \eta_j|$.

   We know that ℓ^1 and ℓ^∞ are metric spaces and that $\ell^1 \subset \ell^\infty$.

   Is ℓ^1 closed in ℓ^∞? If yes, prove it. If not, provide a counterexample.