Analysis Prelim—January 2014

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your work.
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Problems

1. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ have the property that for every $x \in A$, there exists $\varepsilon > 0$ such that $f(t) > \varepsilon$ if $t \in (x - \epsilon, x + \epsilon) \cap A$. Prove that if the set A is compact, there exists $c > 0$ such that $f(x) > c$ for all $x \in A$.

2. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space (X, d). Show that $\{x_n\}$ converges if and only if $\{x_n\}$ has a convergent subsequence.

3. For $f, g : [0, 1] \to \mathbb{R}$, let
$$d(f, g) = \sup_{x \in [0,1]} \{x^2 \, |f(x) - g(x)|\}.$$
   (a) Prove that d defines a metric on the linear space $C([0, 1])$ of continuous functions $f : [0, 1] \to \mathbb{R}$.
   (b) Prove that the resulting metric space $(C([0, 1]), d)$ is not complete.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, i.e., f is periodic. Prove that f is uniformly continuous on $\mathbb{R}$.

5. Suppose that $T : X \to X$ is a continuous mapping of a complete metric space, (X, d), onto itself. Define the n-time composition of T
$$T^n = T \circ T \circ ... \circ T.$$
   Recall that f is a *contraction* mapping if $d\,(f(x),\; f(y)) \leq cd(x, y)$ for some c, $0 \leq c < 1$ for all x, y in the metric space.
   (a) Proof the following statement from Rudin: If T is contracting, then T has a unique fixed point in X.
   (b) Use the statement in (5a) to show: If T^n is contracting for some $n \geq 1$, then T has a unique fixed point in X.

6. Give a proof of the following formulation of the fundamental theorem of calculus:

   Theorem. *Let $F : [a, b] \to \mathbb{R}$ be continuously differentiable, and let $f = \frac{d}{dx}F$. Then*
$$\int_a^b f(x)\; dx = F(b) - F(a).$$

7. The following is a theorem in Rudin:

Theorem. *Suppose that*

i. K *is compact,*

ii. $\{f_n\}$ *is a sequence of continuous functions on* K*,*

iii. $\{f_n\}$ *converges pointwise to a continuous function* f *on* K*,*

iv. $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in K$*,* $n = 1, 2, 3, \ldots$*.*

Then $f_n \to f$ *uniformly.*

Show counterexamples (with justifications!) for the following statements missing or weakening one of the conditions:

(a) Suppose that

   i.

   ii. $\{f_n\}$ is a sequence of continuous functions on K,

   iii. $\{f_n\}$ converges pointwise to a continuous function f on K,

   iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

   Then $f_n \to f$ uniformly.

(b) Suppose that

   i. K is compact,

   ii. $\{f_n\}$ is a sequence of functions on K,

   iii. $\{f_n\}$ converges pointwise to a continuous function f on K,

   iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

   Then $f_n \to f$ uniformly.

(c) Suppose that

   i. K is compact,

   ii. $\{f_n\}$ is a sequence of continuous functions on K,

   iii. $\{f_n\}$ converges pointwise to a function f on K,

   iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

   Then $f_n \to f$ uniformly.

(d) Suppose that

   i. K is compact,

   ii. $\{f_n\}$ is a sequence of continuous functions on K,

   iii. $\{f_n\}$ converges pointwise to a continuous function f on K,

   iv.

   Then $f_n \to f$ uniformly.