Problem 1. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ have the property that for every $x \in A$, there exists $\epsilon > 0$ such that $f(t) > \epsilon$ if $t \in (x - \epsilon, x + \epsilon) \cap A$. If the set A is compact, prove there exists $c > 0$ such that $f(x) > c$ for all $x \in A$.

Proof:

For any $x \in A$, there exists $\epsilon_x > 0$ such that $f(t) > \epsilon_x$ for $t \in (x - \epsilon_x, x + \epsilon_x) \cap A$. The set $\{(x - \epsilon_x, x + \epsilon_x) : x \in A\}$ forms an open cover of A. Since A is compact, there exists a finite subcover that can be written as $\{(x_i - \epsilon_{x_i}, x_i + \epsilon_{x_i}) : 1 \leq i \leq n,\, n \in \mathbb{N}\}$. Choose such a finite subcover.

Let $c = \min\{\epsilon_{x_1}, \epsilon_{x_2}, \ldots, \epsilon_{x_n}\}$.

For all $x \in A$, $x \in (x_i - \epsilon_{x_i}, x_i + \epsilon_{x_i}) \cap A$ for some $i \in \{1, 2, \ldots, n\}$, so $f(x) > \epsilon_{x_i} \geq c$. $\square$

Problem 2. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space (X, d). Show $\{x_n\}$ converges if and only if $\{x_n\}$ has a convergent subsequence.

Proof:

($\Rightarrow$) Suppose $\{x_n\}$ converges. Since $\{x_n\}$ is trivially its own subsequence, $\{x_n\}$ has a convergent subsequence.

($\Leftarrow$) Suppose $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit x.

Consider any $\epsilon > 0$.

Since $\{x_n\}$ is Cauchy, there exists N_1 such that for all $n, m > N_1$, we have that $d(x_n, x_m) < \epsilon/2$. Choose such an N_1.

Since $\{x_{n_k}\}$ converges to x, there exists N_2 such that for all $k > N_2$, we have that $d(x_{n_k}, x) < \epsilon/2$. Choose such an N_2. Note that for $k > N_2$, we have that $n_k > N_2$ by the definition of a subsequence.

Set $N = \max\{N_1, N_2\}$.

For any $n > N$, we have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \text{ where we chose } \textit{any fixed } k > N \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \;\square \end{aligned}$$

Problem 3. For $f, g : [0,1] \to \mathbb{R}$, let

$$d(f,g) = \sup_{x\in[0,1]} \{x^2 \, |f(x) - g(x)|\}.$$

(a) Prove that d defines a metric on the linear space $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

(b) Prove that the resulting metric space $(C([0,1]), d)$ is not complete.

Proof of (a):

- $d(f,g) \geq 0$: a square times an absolute value is non-negative, taking the sup does not change this
- $d(f,g) = 0 \implies f = g$: $d(f,g) = 0$ implies that $f(x) = g(x)$ for all $x \in (0,1]$. As f and g are continuous in 0, this implies that $f(0) = g(0)$.
- $d(f,g) = d(g,f)$: trivial (the definition is symmetric in f and g).
- $d(f,h) \leq d(f,g) + d(g,h)$: By the triangle inequality of $|.|$ in the reals, $x^2|f(x) - h(x)| - x^2|f(x) - g(x)| - x^2|g(x) - h(x)| \leq 0$ for all $x \in [0,1]$. This implies the inequality for the supremum.

Proof of (b): Consider the sequence of functions $f_n(x) = (1-x)^n$. Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0, \end{cases}$$

a function not in $C([0,1])$. On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} d(f_n, 0) &= \lim_{n\to\infty} \sup_{x\in[0,1]} \{x^2 \, |f_n(x)|\} \\
&= \lim_{n\to\infty} \sup_{x\in[0,1]} \{x^2 (1-x)^n\} \\
&\leq \lim_{n\to\infty} \sup_{x\in(0,1)} \{x^2 (1-x)^n\} \text{ (as } x^2(1-x)^n = 0 \text{ for } x \in \{0,1\}) \\
&\leq \lim_{n\to\infty} \sup_{x\in(0,1)} \{(1-x)^n\} \\
&= 0,
\end{aligned}$$

so f_n is Cauchy.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, i.e., f is periodic. Prove f is uniformly continuous on $\mathbb{R}$.

Proof:

Let $A = [0, T] \subset \mathbb{R}$, $B = [T, 2T] \subset \mathbb{R}$ and $C = A \cup B$. Since A, B, and C are compact, f is uniformly continuous on A, B, and C (standard theorem: continuous functions on compact sets are uniformly continuous).

Thus, for any $\epsilon > 0$, there exists $\delta_C > 0$ such that for all $a, b \in C$ with $|a - b| < \delta_C$ implies $|f(a) - f(b)| < \epsilon$.

Set $\delta = \min\{\delta_C, T/2\}$. (Note: You can use T instead of $T/2$.)

Consider any $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Without loss of generality assume that $x \leq y$.

Observe that there exists $k \in \mathbb{Z}$ such that $x = a + kT$ and $y = b + kT$ for some $a, b \in C$. (Note: It looks like there are two cases to consider, (I) $a, b \in A$ or (II) $a \in A$ and $b \in B$, but the proof for each is exactly the same by construction. Also, if we had not made sure that δ was bounded by T then we cannot necessarily state that the *same* $k \in \mathbb{Z}$ can be used to represent $x = a + kT$ and $y = b + kT$, which means we cannot necessarily get to $|a - b| < \delta$ below.)

Since f is periodic, $f(x) = f(a)$ and $f(y) = f(b)$. Also, $|x - y| = |a - b| < \delta \leq \delta_C$ so we have that $|f(x) - f(y)| = |f(a) - f(b)| < \epsilon$. $\square$

Problem 5. [...]

Proof of (a):

Pick any arbitrary but fixed $x_0 \in X$, and define $\{x_n\}$ recursively as $x_{n+1} = T(x_n)$ for $n \in \mathbb{N}$. For $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq cd(x_n, x_{n-1}).$$

By induction, for any $n \in \{0, 1, 2, \ldots\}$,

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0).$$

If $n < m$, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \\ &\leq d(x_1, x_0) \sum_{i=n+1}^{m} c^i \\ &\leq d(x_1, x_0) \frac{c^n}{1-c}. \end{aligned}$$

Since $c < 1$, $c^n \to 0$ as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists some $x \in X$ that is the limit of $\{x_n\}$. Since T is a contraction, it is a continuous function, so

$$T(x) = \lim T(x_n) = \lim x_{n+1} = x.$$

If $y \in X$ is any other fixed point, then $d(x, y) \leq cd(x, y)$, which implies $d(x, y) = 0$. Thus, the fixed point is unique. $\square$.

Proof of (b):

Since T^n is a contraction, by part (a), there exists unique fixed point x of T^n. We show this is the unique fixed point of T. Observe that

$$T^n(Tx) = T^{n+1}(x) = T(T^{n+1}x) = Tx.$$

Thus, Tx is a fixed point of T^n, and by uniqueness, $Tx = x$. Thus, x is a fixed point of T. Since any fixed point of T is also a fixed point of T^n, which is unique, x is the unique fixed point of T. $\square$

Problem 6. Give a proof of the following formulation of the fundamental theorem of calculus:

Let $F : [a, b] \to \mathbb{R}$ be continuously differentiable, and let $f = \frac{d}{dx}F$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Proof: Rudin 6.21

Problem 7. [...]

7(a): $f_n(x) = 1/(nx+1)$ for $x \in (0,1)$. Then f_n are clearly continuous, $f_n \to 0$ monotonically, but convergence is not uniform (the problem occurs near $x = 0$ - students should show).

7(b): $f_n(x) = x^n$ for $x \in [0,1)$ and $f_n(1) = 0$ for all $n \in \mathbb{N}$. Then $f_n \to 0$ monotonically on $[0,1]$, but convergence is not uniform (the problem occurs near $x = 1$ - students should show).

7(c): $f_n(x) = x^n$ for $x \in [0,1]$. Same problem as in (b).

7(d): $f_n(x) = n^2x(1-x^2)^n$ for $x \in [0,1]$. Then $f_n \to 0$. Convergence cannot be uniform since the limit of the integral is not the integral of the limit on $[0,1]$.