University of Colorado Denver
Department of Mathematical and Statistical Sciences

Applied Analysis Preliminary Exam
June 7, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best.
- Submit **no more than** 6 solutions. If you submit more than 6 solutions only the first six problems (as determined by the numbering of the problems) will be graded.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let $x_1 = 1$ and $x_n = \sqrt{3 + \sqrt{x_{n-1}}}$, $n > 1$. Prove that $\{x_n\}$ converges.

2. Prove that Cauchy sequences converge.

3. Prove that if $\{f_n\}$ is a sequence of Riemann integrable functions, and $f_n \to f$ uniformly on $[a, b]$ then f is Riemann integrable on $[a, b]$.

4. Let $f(x)$ be continuously differentiable with $f(0) < -1$, $f(1) > 0$, and $f(2) < 0$. Prove that $\forall c \in [0, 1]$, $\exists x_c \in (0, 2)$ with $f'(x_c) = c$.

5. Prove that

   (a) $\sum_{n=1}^{\infty} \sin(\frac{1}{n}) = \infty$

   (b) $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2}) < \infty$

6. Let $F : \Re^2 \to \Re^2$ be $F(x, y) = (x + y, x^2 + y^2)$.

   (a) Find $A = \{(x, y) \in \Re^2 \ : \ F \text{ is not locally invertable at } (x, y)\}$. Demonstrate that F is not one-to-one in any neighborhood of A.

   (b) Find a first order approximation of F at $(x, y) \in \Re^2$. Is the approximation valid on A?
   (Note: A first order approximation of F at (x, y) is an affine function G such that $\|F(u, v) - G(u, v)\| \sim o(\|(x, y) - (u, v)\|)$.

7. Let $\Re^\infty$ be the space of sequences, $\{x_1, x_2, \ldots\}$, $x_n \in \Re$, and define

$$H = \{(x_1, x_2, \ldots) \in \Re^\infty : \ \sum x_i^2 < \infty\}$$

$$G_n = \{(x_1, x_2 \ldots, x_n, 0, 0, \ldots) \in \Re^\infty\}$$

   and

$$G = \cup_{n=1}^{\infty} G_n.$$

   (a) Is $H \subset G$, $G \subset H$, or $G = H$? Explain.

   (b) Prove that G is dense in H in the ℓ^2 metric.

8. Let X and Y be metric spaces, and let $f_n : X \to Y$ be a sequence of continuous functions that converge uniformly to f. Prove that f is continuous.