Spring 2013 Analysis Prelim Solutions

1. Let $x_1 = 1$ and $x_n = \sqrt{3 + \sqrt{x_{n-1}}},\ n > 1$. Prove that $\{x_n\}$ converges.

   Solution: First we show that $\{x_n\}$ is an increasing sequence. $x_2 = 2 > x_1$. Suppose $x_k > x_{k-1},\ k < n$. Then

   $$x_n^2 - x_{n-1}^2 = (3 + \sqrt{x_{n-1}}) - (3 + \sqrt{x_{n-2}}) = \sqrt{x_{n-1}} - \sqrt{x_{n-2}} > 0.$$

   To show that the sequnce is bounded note that $x_1 < 3$, and suppose $x_k < 3,\ k < n$. Then

   $$x_n^2 = 3 + \sqrt{x_{n-1}} \leq 3 + \sqrt{3} < 9,$$

   so $x_n < 3$ as well. Since the sequence is increasing and bounded, it must converge, e.g., Rudin, Theorem 3.14.

2. Prove that Cauchy sequences converge.

   Solution: Standard, e.g., Rudin, Theorem 3.11.

3. Prove that if $\{f_n\}$ is a sequence of Riemann integrable functions, and $f_n \to f$ uniformly on $[a, b]$ then f is Riemann integrable on $[a, b]$.

   Solution: Standard, e.g., Rudin, Theorem 7.16.

4. Let $f(x)$ be continuously differentiable with $f(0) < -1$, $f(1) > 0$, and $f(2) < 0$. Prove that $\forall c \in [0, 1],\ \exists x_c \in (0, 2)$ with $f'(x_c) = c$.

   Solution: By the MVT there is $z_1 \in (0, 1)$ with $f'(z) > 1$. By the MVT, there is $z_2 \in (1, 2)$ with $f'(z_2) < 0$. Since $f'(x)$ is continuous, the IVT implies that for any $c \in (0, 1)$, there is $x_c \in (z_1, z_2)$ with $f'(x) = c$.

5. Prove that

   (a) $\sum_{n=1}^{\infty} \sin(\frac{1}{n}) = \infty$

   (b) $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2}) < \infty$

   Solution: Since

   $$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{\frac{1}{n^2}} = 1$$

   (a) diverges since $\sum \frac{1}{n}$ diverges, and (b) converges since $\sum \frac{1}{n^2}$ converges. (e.g., Corollary 4.3.12, Trench)

6. Let $F : \Re^2 \to \Re^2$ be $F(x, y) = (x + y, x^2 + y^2)$.

   (a) Find $A = \{(x, y) \in \Re^2 \ : \ F$ is not locally invertable at $(x, y)\}$. Demonstrate that F is not one-to-one in any neighborhood of A.

   (b) Find a first order approximation of F at $(x, y) \in \Re^2$. (Note: A first order approximation of F at (x, y) is an affine function G such that $\|F(u, v) - G(u, v)\| \sim o(\|(x, y) - (u, v)\|)$. Is the approximation valid on A?

Solution: (a) The derivative matrix of F is

$$dF = \begin{bmatrix} 1 & 1 \\ 2x & 2y \end{bmatrix}.$$

Since $\det(dF) = 2y - 2x$, F is locally invertable except when $y = x$, so A is the line $y = x$. We can see that F is not one-to-one in any neighborhood of A since $F(x+\epsilon, x-\epsilon) = F(x-\epsilon, x+\epsilon)$ for any $\epsilon > 0$.

(b) Since F is differentiable, it follows (e.g., Trench, Theorem 6.22) that since F is differentiable everywhere, for any (x, y),

$$\lim_{(u,v)\to(x,y)} \frac{F(u, v) - (F(x, y) + dF \begin{bmatrix} u - x \\ v - y \end{bmatrix})}{\|(u, v) - (x, y)\|} = 0.$$

But the numerator in the expression is exactly $F(u, v) - G(u, v)$, where $G(u, v)$ is the first order approximation.

7. Let $\Re^\infty$ be the space of sequences, $\{x_1, x_2, \ldots\}$, $x_n \in \Re$, and define

$$H = \{(x_1, x_2, \ldots) \in \Re^\infty : \ \sum x_i^2 < \infty\}$$

$$G_n = \{(x_1, x_2 \ldots, x_n, 0, 0, \ldots) \in \Re^\infty\}$$

and

$$G = \cup_{n=1}^{\infty} G_n.$$

   (a) Is $H \subset G$, $G \subset H$, or $G = H$? Explain.

   (b) Prove that G is dense in H in the ℓ^2 metric.

Solution: $G \subset H$ since sequences with a finite number of nonzero elements are always in ℓ^2, but there are elements of ℓ^2 with an infinite number of nonzero elements which are therefore in H but not G. To show that G is dense in H, let $h = (h_1, h_2 \ldots) \in H$. Since $\sum h_i^2 < \infty$, for any $\epsilon > 0$ there is N so that $\sum_{i=N+1}^{\infty} h_i^2 < \epsilon$. Thus $g = (h_1, h_2 \ldots, h_N, 0, 0 \ldots) \in G$ and $\|h - g\| < \epsilon$.

8. Let X and Y be metric spaces, and let $f_n : X \to Y$ be a sequence of continuous functions that converge uniformly to f. Prove that f is continuous.

   Solution: Standard, e.g., Rudin, Theorem 7.12.