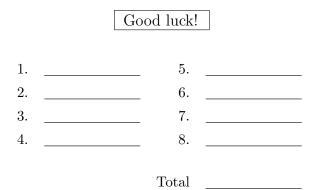
## University of Colorado Denver Deptartment of Mathematical and Statistical Sciences Applied Analysis Preliminary Exam January 18, 2013, 10:00 am – 2:00 pm

## Name:

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in, and only use one side of each sheet.

## Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your <u>six best solutions</u>.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.



DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let  $f_n(x) = (-1)^n \frac{x^n}{n}$ .

- (a) Show that  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly on [0, 1].
- (b) Show that  $\sum_{n=1}^{\infty} |f_n(x)|$  converges pointwise on [0, 1).
- (c) Show that  $\sum_{n=1}^{\infty} |f_n(x)|$  does not converge uniformly on [0, 1).
- 2. Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers, and let  $c_n = a_n + b_n$ .
  - (a) Prove or find a counterexample. If a is a limit point of  $\{a_n\}$  and b is a limit point of  $\{b_n\}$  then a + b is a limit point of  $\{c_n\}$ .
  - (b) Prove or find a counterexample.  $a = \lim a_n$ , and b a limit point of  $\{b_n\}$ , then a + b is a limit point of  $\{c_n\}$ .
- 3. Suppose f(x) is continuous and unbounded on [a, b). Prove that  $\lim_{x\to b^-} f(x)$  does not exist.
- 4. Suppose f is continuous on [a, b]. Prove
  - (a)  $\exists c \in (a, b)$  such that  $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$ .
  - (b)  $\exists c \in (a, b)$  such that  $f(c) = \frac{1}{c-a} \int_a^c f(x) dx$ .
- 5. Suppose  $f : \Re \to \Re$  is continuous with  $\lim_{x\to-\infty} f(x) = \alpha$  and  $\lim_{x\to\infty} f(x) = \beta$ , where  $\alpha, \beta$  are finite. Prove that f is uniformly continuous.
- 6. For which  $\alpha > 0$  does

$$\sum_{k=1}^{\infty} \frac{\alpha^{k \ln k}}{k!}$$

converge?

- 7. Suppose  $f_n \to f$  uniformly on  $A \subset X$ , where (X, d) is a metric space, and let  $x \in \overline{A}$ , i.e., x is a limit point of A. Also assume that for n = 1, 2, 3, ... the limits  $\lim_{t\to x} f_n(t) = a_n \in \Re$  exist.
  - (a) Prove  $\{a_n\}$  converges. Hint: Show the sequence is Cauchy.
  - (b) Prove  $\lim_{t\to x} f(t) = a$ , where  $a = \lim a_n$ .
- 8. Prove that the system of equations

$$3x = y + \sin x$$
$$3y = x + \cos y$$

has a unique solution.