University of Colorado Denver
Deptartment of Mathematical and Statistical Sciences

Applied Analysis Preliminary Exam

January 18, 2013, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in, and only use one side of each sheet.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let $f_n(x) = (-1)^n \frac{x^n}{n}$.

   (a) Show that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$.

   (b) Show that $\sum_{n=1}^{\infty} |f_n(x)|$ converges pointwise on $[0,1)$.

   (c) Show that $\sum_{n=1}^{\infty} |f_n(x)|$ does not converge uniformly on $[0,1)$.

2. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $c_n = a_n + b_n$.

   (a) Prove or find a counterexample. If a is a limit point of $\{a_n\}$ and b is a limit point of $\{b_n\}$ then $a+b$ is a limit point of $\{c_n\}$.

   (b) Prove or find a counterexample. $a = \lim a_n$, and b a limit point of $\{b_n\}$, then $a+b$ is a limit point of $\{c_n\}$.

3. Suppose $f(x)$ is continuous and unbounded on $[a,b)$. Prove that $\lim_{x\to b^-} f(x)$ does not exist.

4. Suppose f is continuous on $[a,b]$. Prove

   (a) $\exists c \in (a,b)$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

   (b) $\exists c \in (a,b)$ such that $f(c) = \frac{1}{c-a}\int_a^c f(x)dx$.

5. Suppose $f : \Re \to \Re$ is continuous with $\lim_{x\to-\infty} f(x) = \alpha$ and $\lim_{x\to\infty} f(x) = \beta$, where α, β are finite. Prove that f is uniformly continuous.

6. For which $\alpha > 0$ does

$$\sum_{k=1}^{\infty} \frac{\alpha^{k \ln k}}{k!}$$

   converge?

7. Suppose $f_n \to f$ uniformly on $A \subset X$, where (X,d) is a metric space, and let $x \in \bar{A}$, i.e., x is a limit point of A. Also assume that for $n = 1,2,3,...$ the limits $\lim_{t\to x} f_n(t) = a_n \in \Re$ exist.

   (a) Prove $\{a_n\}$ converges. Hint: Show the sequence is Cauchy.

   (b) Prove $\lim_{t\to x} f(t) = a$, where $a = \lim a_n$.

8. Prove that the system of equations

$$3x = y + \sin x$$
$$3y = x + \cos y$$

   has a unique solution.