University of Colorado Denver — Dept. Math. & Stat. Sciences

Applied Analysis Preliminary Exam

1 June 2012, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________

Total ____________

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1. Let $n > 1$, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable.

   (a) (10 points) Prove that for all $x \in \mathbb{R}^n$ there is a $p \in \mathbb{R}^n$ such that the derivative in the direction p is zero, i.e., $\frac{\partial f}{\partial p}(x) = 0$.

   (b) (10 points) Let $\alpha \neq 0$. Is there necessarily a unit vector p such that $\frac{\partial f}{\partial p} = \alpha$?

2. Let $\{\mathbf{x}_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$.

   (i) (4 points) Define what it means for $\{\mathbf{x}_i\}_{i=1}^{\infty}$ to be a Cauchy sequence.

   (ii) (10 points)Show that if $\{\mathbf{x}_i\}_{i=1}^{\infty}$ is a convergent sequence then it is a Cauchy sequence.

   (iii) (6 points) Show that if $\{\mathbf{x}_i\}_{i=1}^{\infty}$ is a Cauchy sequence then it is bounded.

3. Let (X, d) be a metric space, and let $B_r(x) = \{y \in X\colon d(x, y) < r\}$.

   (a) (12 points) Let $X = \mathbb{R}^n$, and let $x, y \in X$. Prove that if $B_r(x) \subset B_s(y)$, then $d(x, y) \leq s - r$.

   (b) (8 points) Is (a) true if X is a general metric space? Prove or find a counterexample.

4. Prove that if $|f'(x)| < M$ for all $x \in \mathbb{R}$, then for any $a, b \in \mathbb{R}$,

$$\int_a^b f(x)dx - (b - a)f(a) < \frac{1}{2}M(b - a)^2.$$

5. Let $f(x),\ x \in \mathbb{R}$ be continuous.

   (a) (15 points) Suppose that for all $(a, b) \subset \mathbb{R}$ there is a $z \in (a, b)$ such that $|f(z)| < b - a$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) (5 points) Can you prove (a) without assuming that f is continous? Prove or find a counterexample.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous on the closed interval $[a, b]$, where $a < b$. Let G be the graph of f in $\mathbb{R}^2$, i.e.,

$$G = \{(x, y) \in \mathbb{R}^2\colon y = f(x),\ a \leq x \leq b\}.$$

   Show that for each $\epsilon > 0$ there is a finite set $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ of rectangles that cover G and for which

$$\sum_{j=1}^{m} \text{Area}(T_j) < \epsilon.$$

7. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers bounded above and let $\{\ell_\alpha\}_{\alpha\in\mathcal{I}}$ be a nonempty set of limit points of $\{x_n\}$. Let $\ell = \sup\{\ell_\alpha\}_{\alpha\in\mathcal{I}}$. Prove that ℓ is a limit point of $\{x_n\}$.

8. In this problem we introduce the concepts of lim sup and lim inf for sequences of subsets of $\mathbb{R}^n$, i.e., for collections of subsets of $\mathbb{R}^n$ indexed by the natural numbers. Specifically, let $\{A_k\}_{k=1}^{\infty}$ be a collection of subsets of $\mathbb{R}^n$. Define

$$\limsup\{A_k\} = \cap_{j=1}^{\infty}\left(\cup_{k=j}^{\infty} A_k\right)$$

$$\liminf\{A_k\} = \cup_{j=1}^{\infty}\left(\cap_{k=j}^{\infty} A_k\right).$$

   (a) (8 points) Prove that $\limsup\{A_k\} = \{x \in \mathbb{R}^n : x \in A_k \text{ for infinitely many } k\}$.

   (b) (4 points) State an analogous result for $\liminf\{A_k\}$.

   (c) (8 points) Prove that $\liminf\{A_k\} \subset \limsup\{A_k\}$.