University of Colorado Denver — Dept. Math. & Stat. Sciences

Applied Analysis Preliminary Exam with Solutions

1 June 2012, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

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1. Let $n > 1$, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable.

   (a) (10 points) Prove that for all $x \in \mathbb{R}^n$ there is a $p \in \mathbb{R}^n$ such that the derivative in the direction p is zero, i.e., $\frac{\partial f}{\partial p}(x) = 0$.

   (b) (10 points) Let $\alpha \neq 0$. Is there necessarily a unit vector p such that $\frac{\partial f}{\partial p} = \alpha$?

   Solution: (a) $\frac{\partial f}{\partial p}(x) = \nabla f(x) \cdot p$, so any p that is orthogonal to $\nabla f(x)$ will work. Since $n > 1$ there are $n-1 > 0$ linearly independent directions in which the directional derivative is zero.

   (b) The function defined by $f(x) = 0$ for all $x \in \mathbb{R}^n$ is an easy counterexample.

2. Let $\{\mathbf{x}_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$.

   (i) (4 points) Define what it means for $\{\mathbf{x}_i\}_{i=1}^{\infty}$ to be a Cauchy sequence.

   (ii) (10 points)Show that if $\{\mathbf{x}_i\}_{i=1}^{\infty}$ is a convergent sequence then it is a Cauchy sequence.

   (iii) (6 points) Show that if $\{\mathbf{x}_i\}_{i=1}^{\infty}$ is a Cauchy sequence then it is bounded.

   Solution: See any text for Intro to Real Analysis

3. Let (X, d) be a metric space, and let $B_r(x) = \{y \in X \colon d(x, y) < r\}$.

   (a) (12 points) Let $X = \mathbb{R}^n$, and let $x, y \in X$. Prove that if $B_r(x) \subset B_s(y)$, then $d(x, y) \leq s - r$.

   (b) (8 points) Is (a) true if X is a general metric space? Prove or find a counterexample.

   Solution:

   (a) Replace X by the line passing through x and y, and without loss of generality let $X = R^1$. Then $B_r(x) = (x - r, x + r)$ and $B_s(y) = (y - s, y + s)$. Consider the case when $x < y$. Since $B_r(x) \subset B_s(y)$, we have $y - s \leq x - r$, giving $d(x, y) = y - x \leq s - r$. In the case when $x > y$, $y + s \geq x + r$ gives $s - r \geq x - y = d(x, y)$.

   (b) Counterexample: $X = (0, 1)$, $x = 1/3$, $y = 2/3$, $r = s = 1$. Then $B_r(x) = B_s(y) = X$ but $d(x, y) = 1/3 > s - r = 0$

   Alternative solution to part (a) set in $\mathbb{R}^n$.

   Suppose that $B_r(x) \subseteq B_s(y)$. We want to show that $d(x, y) \leq s - r$. Clearly x must be in $B_s(y)$, so $d(x, y) < s$. Our proof is by contradiction. So we suppose that $d(x, y) > s - r$. This means we can find a small $\epsilon > 0$ such that $0 < \epsilon < r$ and $d(x, y) \geq s - r + 2\epsilon$.

   On the line through x and y we carefully choose a point z such that x is between y and z with $d(x, z)$ slightly less than r (so $z \in B_r(x)$), and then we show that $d(y, z) > s$, i.e., $B_r(x) \not\subset B_s(y)$. Hence we want to find a z of the form $z = y + t(x - y)$ with $t > 1$ (so x is between y and z). Put

$$t = \frac{s+\epsilon}{d(y,x)} > 1 \text{ since } d(x,y) < s + \epsilon,$$

so $d(y,z) = \|y - z\| = t \cdot d(y,x) = s + \epsilon > s$. This shows $z \notin B_s(y)$. To see that $z \in B_r(x)$, compute

$$d(x,z) = \|x - z\| = \|x - y - t(x-y)\| = |t-1| \cdot \|y - x\|.$$

$$= (t-1) \cdot d(y,x) = \frac{s + \epsilon - d(y,x)}{d(y,x)} \cdot d(y,x) = s + \epsilon - d(y,x) \leq (s+\epsilon) - (s - r + 2\epsilon) = r - \epsilon.$$

So we see $z \in B_r(x) \setminus B_s(y)$, a contradiction that completes the proof.

4. Prove that if $|f'(x)| < M$ for all $x \in \mathbb{R}$, then for any $a, b \in \mathbb{R}$,

$$\int_a^b f(x)dx - (b-a)f(a) < \frac{1}{2}M(b-a)^2.$$

Solution:

$$\int_a^b f(x)dx - (b-a)f(a) = \int_a^b (f(x) - f(a))dx \leq \int_a^b |f(x) - f(a)|dx.$$

The Mean Value theorem implies that $|f(x) - f(a)| \leq M(x-a)$, so

$$\int_a^b |f(x) - f(a)|dx \leq \int_a^b M(x-a)dx = \frac{1}{2}M(b-a)^2.$$

5. Let $f(x),\ x \in \mathbb{R}$ be continuous.

   (a) (15 points) Suppose that for all $(a,b) \subset \mathbb{R}$ there is a $z \in (a,b)$ such that $|f(z)| < b - a$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) (5 points) Can you prove (a) without assuming that f is continous? Prove or find a counterexample.

Solution: (a) Suppose that for some x $f(x) > 0$. Since f is continuous, there exists a $\delta > 0$ such that if $|y - x| < \delta$ then $f(y) > f(x)/2$. Choose $z_1, z_2 \in (x - \delta, x + \delta)$ so that $z_1 < z_2$ and $|z_1 - z_2| < f(x)/2$. Then by hypothesis there is a point $z_3 \in (z_1, z_2)$ such that $f(z_3) < z_2 - z_1 < f(x)/2$. But $z_3 \in (x - \delta, x + \delta)$, so $f(z_3) > f(x)/2$, which is a contradiction. If $f(x) \leq 0$ for all $x \in \mathbb{R}$, replace f with $-f$ to reach the same contradiction.

(b) False, for example $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous on the closed interval $[a, b]$, where $a < b$. Let G be the graph of f in $\mathbb{R}^2$, i.e.,

$$G = \{(x, y) \in \mathbb{R}^2 \colon y = f(x),\ a \le x \le b\}.$$

Show that for each $\epsilon > 0$ there is a finite set $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ of rectangles that cover G and for which

$$\sum_{j=1}^{m} \text{Area}(T_j) < \epsilon.$$

Solution: Let $\epsilon > 0$ be given. Since f is continuous on the compact set $[a, b]$ it must be uniformly continuous there. So we may choose a $\delta > 0$ such that

$$|f(x') - f(x'')| < \frac{\epsilon}{3(b-a)} \text{ if } x', x'' \in [a, b] \text{ and } |x' - x''| < \delta.$$

Let $P\colon a = x_0 < x_1 < \cdots < x_n = b$ be a partion of $[a, b]$ with $x_i - x_{i-1} < \delta$ for all $i = 1, 2, \ldots, n$. Choose $\zeta_1, \ldots, \zeta_n$ such that

$$x_{i-1} \le \zeta_i \le x_i, \quad 1 \le i \le n.$$

Then by the choice of δ we have

$$|f(x) - f(\zeta_i)| < \frac{\epsilon}{3(b-a)} \text{ if } x_{i-1} \le x \le x_i.$$

This means that every point on the graph G above the interval $[x_{i-1}, x_i]$ is in a rectangle $[x_{i-1}, x_i] \times [f(\zeta_i) - \frac{\epsilon}{3(b-a)}, f(\zeta_i) + \frac{\epsilon}{3(b-a)}]$ with area $2\frac{\epsilon}{3(b-a)}(x_i - x_{i-1})$. Since the total area of these rectangles is $\frac{2}{3}\epsilon < \epsilon$, the graph has zero content.

7. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers bounded above and let $\{\ell_\alpha\}_{\alpha \in \mathcal{I}}$ be a nonempty set of limit points of $\{x_n\}$. Let $\ell = \sup\{\ell_\alpha\}_{\alpha \in \mathcal{I}}$. Prove that ℓ is a limit point of $\{x_n\}$.

Solution: Since the sequence $\{x_n\}$ is bounded above, say by M, then clearly also $\{\ell_\alpha\}_{\alpha \in \mathcal{I}}$ must be bounded above by M, so that $\ell = \sup\{\ell_\alpha\}_{\alpha \in \mathcal{I}}$ is a real number. We will construct a subsequence x_{n_i} such that $|x_{n_i} - \ell| < 1/i$. Fix $i > 0$ and suppose $x_{n_1}, \ldots x_{n_{i-1}}$ were already chosen. Since $\ell = \sup\{\ell_\alpha\}$, there exists some ℓ_β such that $\ell \ge \ell_\beta > \ell - 1/2i$. Since ℓ_β is a limit point of $\{x_\alpha\}$, there exists $n_i > n_{i-1}$ such that $|x_{n_i} - \ell_\beta| < 1/2i$, where we put $n_0 = 0$ for $i = 1$. Then

$$|x_{n_i} - \ell| \le |x_{n_i} - \ell_\beta| + |\ell_\beta - \ell| < 1/2i + 1/2i = 1/i.$$

8. In this problem we introduce the concepts of lim sup and lim inf for sequences of subsets of $\mathbb{R}^n$, i.e., for collections of subsets of $\mathbb{R}^n$ indexed by the natural numbers. Specifically, let $\{A_k\}_{k=1}^{\infty}$ be a collection of subsets of $\mathbb{R}^n$. Define

$$\limsup\{A_k\} = \cap_{j=1}^{\infty} \left(\cup_{k=j}^{\infty} A_k\right)$$

$$\liminf\{A_k\} = \cup_{j=1}^{\infty} \left(\cap_{k=j}^{\infty} A_k\right).$$

(a) (8 points) Prove that $\limsup\{A_k\} = \{x \in \mathbb{R}^n : x \in A_k \text{ for infinitely many } k\}$.

(b) (4 points) State an analogous result for $\liminf\{A_k\}$.

(c) (8 points) Prove that $\liminf\{A_k\} \subset \limsup\{A_k\}$.

Solution: (a) If $x \in \limsup\{A_k\}$ then for each j there is a $k \geq j$ such that $x \in A_k$. Suppose that $\{k : x \in A_k\}$ is finite. Let k^* be the largest element of the set. But the first line of this proof implies there is a $k > k^*$ such that $x \in A_k$, and this contradicts the definition of k^*. Conversely, suppose that $x \in A_k$ for infinitely many values of k. Then for each j there is a $k \geq j$ such that $x \in A_k$. It follows immediately by the definition of lim sup that $x \in \limsup\{A_k\}$.

(b) $\liminf\{A_k\} = \{x \in \mathbb{R}^n : x \in A_k \text{ for all but a finite number of } k\}$.

(c) If $x \in \liminf\{A_k\}$, then there is a k^* such that for all $k > k^*$, $x \in A_k$, so $x \in A_k$ for infinitely many k, i.e., $x \in \limsup\{A_k\}$.