University of Colorado Denver — Dept. Math. & Stat. Sciences

Applied Analysis Preliminary Exam

13 January 2012, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let a and b be real numbers with $a \neq 0$. Determine the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(ax)^n}{\left(1+\frac{b}{n}\right)^{n^2}}.$$

Solution: This is a power series so we use the root test on the series of absolute values.

$$\lim_{n\to\infty}\left(\frac{|ax|^n}{\left(1+\frac{b}{n}\right)^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{|ax|}{\left(1+\frac{b}{n}\right)^{n}} = \frac{|ax|}{e^b} < 1 \iff |x| < \frac{e^b}{|a|}.$$

Hence the radius of convergence is $\frac{e^b}{|a|}$.

2. Given two real, bounded sequences $\{a_n\}$ and $\{b_n\}$, put $\overline{a} = \limsup_{n\to\infty}\{a_n\}$ and $\overline{b} = \limsup_{n\to\infty}\{b_n\}$. Also, put $\overline{s} = \limsup\{a_n + b_n\}$. Prove that

$$\overline{s} \leq \overline{a} + \overline{b} \text{ when it is valid.}$$

Is it always valid?

Since both sequences are bounded, we clearly have that $\overline{a} < +\infty$ and $\overline{b} < +\infty$. Choose real numbers a and b such that $\overline{a} < a$ and $\overline{b} < b$. Since $a > \overline{a}$, we know that a is eventually an upper bound, i.e., there is an integer N_a for which $a_n \leq a$ for $n \geq N_a$. Similarly, there is an integer N_b for which $b_n \leq b$ for $n \geq N_b$. Put $N = \max\{N_a, N_b\}$. Then for $n \geq N$ we have $a_n + b_n \leq a + b$. Hence $a + b$ is eventually an upper bound, which implies that

$$\overline{s} = \limsup_{n\to\infty}\{a_n + b_n\} \leq a + b.$$

Since this holds for all $a > \overline{a}$ and all $b > \overline{b}$, it must be that $\overline{s} \leq \overline{a} + \overline{b}$. In detail we have the following. Suppose that $\overline{s} = \overline{a} + \overline{b} + \epsilon$, where $\epsilon > 0$, then just put $a = \overline{a} + \epsilon/3$ and $b = \overline{b} + \epsilon/3$ in the argument above to see that $\overline{s} \leq \overline{a} + \overline{b} + 2\epsilon/3 < \overline{a} + \overline{b} + \epsilon = \overline{s}$, an impossibility.

3. A real number x is said to be a **dyadic rational** provided there is an integer k and a nonnegative integer n for which $x = \frac{k}{2^n}$. For each $x \in [0,1]$ and each $n \in \mathbb{N}$, put

$$f_n(x) = \begin{cases} 1, & \text{if } x = \frac{k}{2^n} \text{ for some } k \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

   (i) Prove that the dyadic rationals are dense in $\mathbb{R}$.

   Solution: Let $a, b \in \mathbb{R}$ with $a < b$. We want to show that there is a dyadic rational in the open interval (a, b). By the Archimedean principle we can choose a positive integer n such that $2^n(b-a) > 1$, so that we know there is an integer k with $2^n a < k < 2^n b$, hence $a < \frac{k}{2^n} < b$, as desired.

(ii) Let $f\colon [0,1] \to \mathbb{R}$ be the function to which the sequence $\{f_n\}$ converges pointwise. Show that $\int_0^1 f$ does not exist.

Solution: If $x = \frac{k}{2^n} \in [0,1]$ for some $k, n \in \mathbb{N}$, then $x = \frac{k2^r}{2^{n+r}}$ for all $r \geq 0$. This says that $f_m(x) = 1$ for all $m \geq n$. And if x is not a dyadic rational, then $f_n(x) = 0$ for all $n \in \mathbb{N}$. Hence the limit function f is given by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a nonzero dyadic rational;} \\ 0, & \text{otherwise.} \end{cases}$$

Consider any partition $P = \{x_0, \ldots, x_n\}$ of $[0,1]$. By the density of the dyadic rationals $\sup\{f(x)\colon x \in [x_{i-1}, x_i]\} = 1$ for each subinterval of the partition. This says that upper integral of f on $[0,1]$ equals 1. Similarly, since the irrationals are dense in $[0,1]$, it follows that the lower integral of f on $[0.1]$ equals 0. Hence

$$\int_0^1 f \quad \text{does not exist.}$$

(iii) Show that the convergence $\{f_n\} \to f$ is not uniform.

Solution: There are (at least) two ways to prove this. Our first way is the more simplistic. In order to show that the convergence is NOT uniform, we will show that for fixed $\epsilon = \frac{1}{2}$, no matter how large an n we consider, there is some $x_n \in [0.1]$ for which $|f(x_n) - f_n(x_n)| > \frac{1}{2}$. For given n put

$$x_n = \frac{1}{2^{n+1}}.$$

Then x_n is a dyadic rational, so $f(x_n) = 1$. But x_n cannot be written in the form $\frac{k}{2^n}$ for any integer k, so $f_n(x_n) = 0$, implying $|f(x_n) - f_n(x_n)| = 1 > \frac{1}{2}$. Hence the convergence is not uniform.

Our second way to prove this is to quote the theorem that says that if $\{f_n\}$ is a sequence of integrable functions from $[a,b]$ to $\mathbb{R}$, and if $\{f_n\}$ converges uniformly to f, then $\int_a^b f$ exists (and even equals the limit of the integrals). Each f_n has the value 1 at only finitely many values of x, so f_n is integrable on $[0,1]$. As f is not integrable on $[0,1]$, the convergence could not have been uniform.

4. Show that if $\{f_n\}$ and $\{g_n\}$ are two sequences of real bounded functions that converge uniformly on some set $E \subseteq \mathbb{R}$, then

(i) the functions f_n and g_n are uniformly bounded;

(ii) $\{f_n g_n\}$ also converges uniformly on E.

Solution From the Cauchy criterion for uniformly convergent sequences of functions there is an N such that

$$m, n > N \implies \forall x \in E \text{ we have } |f_m(x) - f_n(x)| < 1.$$

Since each f_n is bounded, for each $k \in \mathbb{N}$ and for all $x \in E$ we have $|f_k(x)| \leq c_k$ for some constant c_k. Then using the triangle inequality we see that for $n > N$ we have

$$|f_n(x)| \leq |f_n(x) - f_{N+1}(x)| + |f_{N+1}(x)| < c_{N+1} + 1.$$

Hence

$$\forall n > N \text{ and } \forall k \in \mathbb{N} \text{ and } \forall x \in E \text{ we have } |f_k(x)| \leq \max\{c_1, \ldots, c_N, c_{N+1} + 1\}.$$

This shows that the f_n are uniformly bounded. The same argument works for the functions g_n. So we know there is a constant C such that

$$\forall k \in \mathbb{N} \text{ and } \forall x \in E \text{ we have } |f_k(x)| \leq C \text{ and } |g_k(x)| \leq C.$$

This proves part (i).

We now show that $\{f_n g_n\}$ satisfies the uniform Cauchy criterion. Let $\epsilon > 0$ be given. From the uniform Cauchy criterion for $\{f_n\}$ and $\{g_n\}$, there are N_1 and N_2 such that

$$m, n > N_1 \implies \forall x \in E: \quad |f_m(x) - f_n(x)| < \frac{\epsilon}{2C};$$
$$m, n > N_1 \implies \forall x \in E: \quad |g_m(x) - g_n(x)| < \frac{\epsilon}{2C}.$$

Now let $m, n > N = \max\{N_1, N_2\}$ and compute that

$$\begin{aligned}
& |f_m(x)g_m(x) - f_n(x)g_n(x)| \\
\leq \; & |f_m(x)g_m(x) - f_n(x)g_m(x)| + |f_n(x)g_m(x) - f_n(x)g_n(x)| \\
= \; & |f_m(x) - f_n(x)| \cdot |g_m(x)| + |f_n(x)| \cdot |g_m(x) - g_n(x)| \\
< \; & \frac{\epsilon}{2C}C + \frac{\epsilon}{2C}C = \epsilon.
\end{aligned}$$

5. Let A be a subset of the metric space (X, d). Suppose that x is a limit point of $\overline{A}$. Show that x must be a limit point of A.

   Solution Suppose that x is a limit point of $\overline{A}$ but not a limit point of A. Since x is not a limit point of A there must be an open ball $B_r(x)$, $r > 0$, with $B_r(x) \cap A = \emptyset$. But since x is a limit point of $\overline{A}$, each open set containing x must contain a point y of $\overline{A} \setminus \{x\}$. But since $y \in \overline{A}$, each open set containing y must have a point of A. This means $B_r(x)$ must contain a point of A, giving a contradiction. Hence x must be a limit point of A.

   Alternative Solution: We have seen that $\overline{A}$ is closed, and that a closed set contains all its limit points. Hence $x \in \overline{A}$. We suppose that x is NOT a limit point of A and work for a contradiction. It follows that $x \in A$ and is not a limit point of A. But since x is a limit point of $\overline{A}$, for each $n \geq 1$ there is a point $y_n \in B_{\frac{1}{2n}}(x) \cap (\overline{A} \setminus \{x\})$.

   Put $r_n = d(y_n, x)$, so $0 < r_n < \frac{1}{2n}$. Since x is not a limit point of A, there must be some $r > 0$ such that $B_r(x) \cap A = \{x\}$. Choose N large enough so that $n > N \implies \frac{1}{2n} < r$. Then $n > N \implies y_n \in \overline{A} \setminus A \implies y_n$ is a limit point of A. Hence $B_{r_n}(y_n)$ contains a point $x_n \in A$. But $d(x_n, y_n) < r_n = d(y_n, x) \implies x_n \neq x$. And

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) < r_n + r_n < \frac{1}{n}.$$

We have now established a sequence $\{x_n\}$ of points of A different from x with $d(x_n, x) < \frac{1}{n}$ for all $n > N$.

Let U be any open set containing x. There must be some $s > 0$ such that $x \in B_s(x) \subseteq U$. Now pick $N' > N$ such that $n > N' \implies \frac{1}{n} < s$. Then $n > N'$ implies that

$$x_n \in B_{\frac{1}{n}}(x) \subseteq B_s(x) \subseteq U, \ x_n \in A, \ x_n \neq x.$$

This shows that x is a limit point of A, contradicting our hypothesis that x is not a limit point of A. Hence it must be that x is indeed a limit point of A.

6. Let (X, d) and (Y, ρ) be metric spaces and let $\emptyset \neq A \subseteq X$. If f and g are continuous mappings of X into Y for which $f(x) = g(x)$ for all $x \in A$, show that $f(x) = g(x)$ for all $x \in \overline{A}$.

   Solution: Let $x \in \overline{A} \setminus A$. We want to show that $f(x) = g(x)$. Let $\epsilon > 0$ be given. Since f and g are continuous at x, there are $\delta_g > 0$ and $\delta_f > 0$ such that the following hold:

   $$d(x, y) < \delta_f \implies \rho(f(x), f(y)) < \epsilon/2;$$

   and

   $$d(x, y) < \delta_g \implies \rho(g(x), g(y)) < \epsilon/2.$$

   Since x is a limit point of A by hypothesis, for each positive integer n, the open ball $B_{1/n}(x)$ contains a point $a_n \in A$. Choose n large enough so that $\frac{1}{n} < \min\{\delta_f, \delta_g\}$. Then we have

   $$\rho(f(x), g(x)) \leq \rho(f(x), f(a_n)) + \rho(f(a_n), g(a_n)) + \rho(g(a_n), g(x))$$

   $$\leq \epsilon/2 + 0 + \epsilon/2 = \epsilon.$$

   Since $\epsilon > 0$ is arbitrary, we must have $f(x) = g(x)$.

7. Find

   $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}},$$

   and justify your answer. Hint: Think of the sum as a Riemann sum.

   Solution: For each n the sum is a Riemann sum, $\sum_{i=1}^{n} f(t_i)\Delta x_i$, where $t_i = i/n$ and the partition is uniform, i.e., $\Delta x_i = 1/n$. By Theorem 6.14 in Rudin (or any such theorem), the sum converges to $\int_0^1 \sqrt{x}dx = 2/3$.

8. Let $f(x)$ be continuous, real-valued function on $[a, b]$.

   (a) Prove that for any $\epsilon > 0$ there is a polynomial, $p(x)$, such that

   $$\int_a^b |f(x) - p(x)|dx < \epsilon.$$

Solution: By the Weierstrass theorem (e.g. thm 7.24 Rudin), there exists $p(x)$ so that for every $x \in [a, b]$, $|f(x) - p(x)| < \epsilon/(b - a)$. So,

$$\int_a^b |f(x) - p(x)|dx < \int_a^b \epsilon/(b-a)dx = \epsilon.$$

(b) Prove that there is a sequence $p_k(x)$ of polynomials having the property that

$$f(x) = \sum_{k=1}^{\infty} p_k(x) \text{ for all } x \in [a, b].$$

Solution: By the Weierstrass Theorem, for $\epsilon = 1/2$ there is a polynomial $p_1(x)$ such that for each $x \in [a, b]$ there holds $|f(x)-p_1(x)| < 1/2$. Then for $\epsilon = 1/(2^2)$, there is a polynomial $p_2(x)$ such that for all $x \in [a, b]$ there holds $|(f(x)-p_1(x))-p_2(x)| < 1/(2^2)$. We now proceed by induction. Suppose that $p_1(x), \ldots, p_j(x)$ have been chosen so that

$$|f(x) - (p_1(x) + \cdots + p_k(x))| < \frac{1}{2^k}, \text{ for } 1 \leq k \leq j.$$

Then by the Weierstrass Theorem there is a polynomial $p_j(x)$ such that for all $x \in [a, b]$ we have

$$|f(x) - (p_1(x) + \cdots + p_j(x)) - p_{j+1}(x)| < \frac{1}{2^{j+1}}.$$

By induction we may assume that $p_n(x)$ has been constructed for each $n \in \mathbb{N}$ so that

$$|f(x) - \sum_{i=1}^{m} p_i(x)| < \frac{1}{2^m}.$$

Hence $f(x) = \sum_{k=1}^{\infty} p_k(x)$ for all $x \in [a, b]$.