University of Colorado Denver — Dept. Math. & Stat. Sciences

Applied Analysis Preliminary Exam

14 January 2011, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$. Define $f\colon \mathbb{R}^* \to [-1, +1]$ by

$$f\colon x \mapsto \begin{cases} -1, & \text{if } x = -\infty; \\ \frac{x}{1+|x|}, & \text{if } x \in \mathbb{R}; \\ +1, & \text{if } x = +\infty. \end{cases} \tag{1}$$

   (i) (6 points) Show that f maps $\mathbb{R}^*$ onto $[-1, +1]$ and is strictly increasing, so it maps $\mathbb{R}^*$ in a one-to-one fashion.

   (ii) (6 points) Define a function $d\colon \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}\colon (x, y) \mapsto |f(x) - f(y)|$, where f is defined by Eq. 1 . Show that d is a metric on $\mathbb{R}^*$, so that $(\mathbb{R}^*, d)$ is a metric space.

   (iii) (8 points) Show that both $(-2, +\infty)$ and $(-2, +\infty]$ are open balls in the metric space $(\mathbb{R}^*, d)$.

2. Let (X, d) be a compact metric space. Let $f\colon X \to \mathbb{R}$ be continuous. Show that f is uniformly continuous on X.

3. Let (X, d) be a metric space and $f\colon X \to X$ a function on X.

   (i) (5 points) Define what it means for f to be a contraction.

   (ii) (15 points) State and Prove the Contraction Mapping Theorem.

4. (i) (10 points) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ for which $\{f_n\}$ converges uniformly to the function f. Show that $\int_a^b f$ exists and that $\lim_{n\to\infty}(\int_a^b f_n) = \int_a^b f$, i.e., $\lim_{n\to\infty}\left(\int_a^b f_n\right) = \int_a^b(\lim_{n\to\infty} f_n)$.

   (ii) (10 points) Give an example of a sequence $\{f_n\}$ of real-valued functions defined on some closed, bounded interval $[a, b]$ such that $\int_a^b f_n$ exists for every $n \in \mathbb{N}$, such that $\{f_n\} \to f$ pointwise on $[a, b]$, and such that $\int_a^b f$ exists, but $\lim_{n\to\infty}\left(\int_a^b f_n\right) \neq \int_a^b (\lim_{n\to\infty} f_n)$.

5. This problem has two parts.

   (i) (10 points) State Taylor's Theorem for real-valued functions of several variables, giving the version that uses multinomial coefficients.

   (ii) (10 points) **Illustrate its use** to write $f(x, y) = x^2y + x^3 + y^3$ as a polynomial in $(x-1)$ and $(y+1)$ by computing its Taylor expansion about the point $\mathbf{c} = (1, -1)$.

6. Define a function $\overline{F}\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\overline{F}\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1x_3 + 2x_2x_3 + 4 \\ x_1^2x_3 + x_1x_2 + 2 \end{pmatrix}.$$

   (i) (4 points) It should be clear that $\overline{F}$ is differentiable on all of $\mathbb{R}^3$. Compute the Jacobian $J_{\overline{F}}$.

(ii) (4 points) Let $\overline{p}_0 = (2, -3, 1)$. Check to see that $\overline{F}(\overline{p}_0) = \overline{0}$ and evaluate $J_{\overline{F}}(\overline{p}_0)$.

(iii) (4 points) Name the theorem (Call it Theorem A) that guarantees that some pairs of variables from $\{x_1, x_2, x_3\}$ can be written as functions of the third variable. List these pairs.

(iv) (8 points) Choose one of the pairs you selected in part (iii). If i, j, k are 1, 2, 3 in some order, suppose that you pick x_i and x_j to write as a function of x_k. Then Theorem A says that there is a differentiable function $\overline{G}$ that gives x_i and x_j as functions of x_k. Compute the Jacobian matrix for $\overline{G}$ at the special value of x_k determined by the original point $\overline{p}_0$. Now do this for all the other special pair(s) you selected in part (iii).

7. This problem has three parts.

(i) (5 points) Suppose that f is continuous at x_0 and $f'(x)$ exists in a deleted neighborhood of x_0 and $\lim_{x \to x_0} f'(x) = L \in \mathbb{R}$. Show that $f'(x_0)$ exists and $f'(x)$ is continuous at x_0.

(ii) (5 points) For each integer $n \geq 0$ show that $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$.

(iii) (10 points) Define

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is infinitely differentiable on $\mathbb{R}$ and that $f^{(n)}(0) = 0$ for all $n \geq 0$. Use this to show that there is no power series $\sum_{n=0}^{\infty} a_n x^n$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all x in some interval $(-r, r)$ for $r > 0$.

8. Prove the following:

(i) (6 points) If $p > 1$ and $0 < x < 1$, then $1 - px < (1 - x)^p$.

(ii) (6 points) Let $p > 1$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of **positive** numbers for which there is an n_0 such that $a_{n+1}/a_n \leq 1 - p/n$ for all $n \geq n_0$. Then $\sum_{n=1}^{\infty} a_n$ converges. (We will call this the S Test.)

(iii) (8 points) Use the S Test to determine for which $a > 0$, $b > 0$ the following (hypergeometric) series converges:

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)b+2)} + \cdots$$