University of Colorado Denver — Dept. Math. & Stat. Sciences

Applied Analysis Preliminary Exam

14 January 2011, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless noted otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$. Define $f\colon \mathbb{R}^* \to [-1, +1]$ by

$$f\colon x \mapsto \begin{cases} -1, & \text{if } x = -\infty; \\ \frac{x}{1+|x|}, & \text{if } x \in \mathbb{R}; \\ +1, & \text{if } x = +\infty. \end{cases} \tag{1}$$

   (i) (6 points) Show that f maps $\mathbb{R}^*$ onto $[-1, +1]$ and is strictly increasing, so it maps $\mathbb{R}^*$ in a one-to-one fashion.

   (ii) (6 points) Define a function $d\colon \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}\colon (x, y) \mapsto |f(x) - f(y)|$, where f is defined by Eq. 1 . Show that d is a metric on $\mathbb{R}^*$, so that $(\mathbb{R}^*, d)$ is a metric space.

   (iii) (8 points) Show that both $(-2, +\infty)$ and $(-2, +\infty]$ are open balls in the metric space $(\mathbb{R}^*, d)$.

   Solution: It is easy to see that for real x, $\frac{x}{1+|x|}$ lies between -1 and $+1$ but never takes on either value. If $-\infty < x < 0$, then $f(x) = \frac{x}{1-x}$, so $f'(x) = \frac{1}{(1-x)^2} > 0$. $x = 0$ is a special case, but the derivative of f at $x = 0$ is easily computed to equal 1. And for $0 < x < +\infty$, $f'(x) = \frac{1}{(1+x)^2} > 0$. So f is strictly increasing, and hence is one-to-one.

   For part (ii), clearly $d(x, y) = d(y, x) \geq 0$ with equality if and only if $x = y$. The triangle inequality is lengthier but hardly more challenging. One just has to consider three cases.

   For part (iii), first consider $(-2, +\infty)$. We want an $x \in \mathbb{R}^*$ such that $d(-2, x) = d(x, \infty)$. Solving $\frac{x}{1+|x|} - \frac{-2}{3} = 1 - \frac{x}{1+|x|}$ gives $x = \frac{1}{5}$. Then $d(-2, \frac{1}{5}) = \frac{5}{6} = d(\frac{1}{5}, +\infty)$, so that $(-2, +\infty) = B_{5/6}(\frac{1}{5})$. To see $(-2, +\infty]$ as an open ball we just need to choose an $x \in \mathbb{R}$ that is closer to $+\infty$ than it is to -2. Put $x = 1$. Then $d(-2, 1) = \frac{1}{2} - \frac{-2}{3} = \frac{3}{6} + \frac{4}{6} = 3/2$. So

$$(-2, +\infty] = B_{3/2}(1).$$

2. Let (X, d) be a compact metric space. Let $f\colon X \to \mathbb{R}$ be continuous. Show that f is uniformly continuous on X.

   Solution: In this solution I use ρ for the standard metric on $\mathbb{R}$, so it is easy to see how to generalize the result.

   Let $\epsilon > 0$ be given. For each $x \in X$, choose $\delta_x > 0$ such that $d(x, y) < \delta_x$ implies that $\rho(f(x), f(y)) < \epsilon/2$. The open balls $B_{\frac{1}{2}\delta_x}(x)$ cover X. Let the balls with centers $x_1, \ldots, x_n$ be a finite subcover. Put $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$. Now suppose that $d(x, y) < \delta$. The point x is in some ball in the finite subcover. Suppose x is in $B_{\frac{1}{2}\delta_{x_j}}(x_j)$. Then $d(x, x_j) < \frac{1}{2}\delta_{x_j}$, so that

$$d(y, x_j) \leq d(y, x) + d(x, x_j) < \delta + \frac{1}{2}\delta_{x_j} \leq \delta_{x_j}.$$

   By definition of δ_{x_j}, $\rho(f(y), f(x_j)) < \epsilon/2$ and $\rho(f(x_j), f(x)) < \epsilon/2$. Therefore

$$\rho(f(y), f(x)) \leq \rho(f(y), f(x_j)) + \rho(f(x_j), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.,$$

   and the proof is complete.

3. Let (X, d) be a metric space and $f \colon X \to X$ a function on X.

   (i) (5 points) Define what it means for f to be a contraction.

   Solution: There is a constant k satisfying $0 < k < 1$ and for all $x, y \in X$ we have $d(f(x), f(y)) \leq k \cdot d(x, y)$.

   (ii) (15 points) State and Prove the Contraction Mapping Theorem.

Solution: The Contraction Mapping Theorem: If (X, d) is a complete metric space and $f \colon X \to X$ is a contraction, then there is a unique $x_0 \in X$ for which $f(x_0) = x_0$.

Proof. It should be easy to prove that a contraction f on X could have at most one fixed point in X. The hard part is to show that there is such a point.

Pick $x_0 \in X$ arbitrarily. Define x_n recursively for $n \geq 1$ by $x_{n+1} = f(x_n)$, for $n \geq 0$. Choose $c < 1$ for which $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Then for $n \geq 1$ we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq c \cdot d(x_n, x_{n-1}) \leq c \cdot c \cdot d(x_{n-1}, x_{n-2}) \leq \cdots .$$

By induction we have $d(x_{n+1} x_n) \leq c^n \cdot d(x_1, x_0)$ for $n \geq 1$. Suppose $1 \leq n < m$, so

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (c^n + c^{n+1} + \cdots + c^{m-1}) \cdot d(x_0, x_1) < \frac{c^m}{1 - c} \cdot d(x_1, x_0). \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence and must converge to some point $\overline{x}$ since the space X is complete. Then using the continuity of f we have

$$f(\overline{x}) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \overline{x}.$$

□

4. (i) (10 points) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ for which $\{f_n\}$ converges uniformly to the function f. Show that $\int_a^b f$ exists and that $\lim_{n\to\infty}(\int_a^b f_n) = \int_a^b f$, i.e., $\lim_{n\to\infty} \left(\int_a^b f_n\right) = \int_a^b (\lim_{n\to\infty} f_n)$.

   Solution: First, note that since the uniform limit of continuous functions is continuous, the limit function f must be continuous on $[a, b]$ and hence integrable there. Now let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to the function f, we may choose an $N \in \mathbb{N}$ such that whenever $n > N$ and for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{b - a}.$$

   Fix $n > N$. Then

$$\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f - f_n| \leq \left[\frac{\epsilon}{b - a} \right] (b - a) = \epsilon.$$

   It follows that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b (\lim_{n\to\infty} f_n)$.

(ii) (10 points) Give an example of a sequence $\{f_n\}$ of real-valued functions defined on some closed, bounded interval $[a,b]$ such that $\int_a^b f_n$ exists for every $n \in \mathbb{N}$, such that $\{f_n\} \to f$ pointwise on $[a,b]$, and such that $\int_a^b f$ exists, but $\lim_{n\to\infty} \left(\int_a^b f_n\right) \neq \int_a^b (\lim_{n\to\infty} f_n)$.

One Answer: Define $f_n(x)$ for $x \in [0,1]$ as follows:

$$f_n(x) = \begin{cases} 4n^2x, & \text{for } 0 \le x \le \frac{1}{2n}; \\ -4n^2x + 4n, & \text{for } \frac{1}{2n} \le x \le \frac{1}{n}; \\ 0, & \text{for } \frac{1}{n} \le x \le 1. \end{cases}$$

Then $\int_0^1 f_n = 1$ for every n, so $\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} 1 = 1$. On the other hand $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$, so $\int_0^1 (\lim_{n\to\infty} f_n) = \int_0^1 0 = 0$.

5. This problem has two parts.

(i) (10 points) State Taylor's Theorem for real-valued functions of several variables, giving the version that uses multinomial coefficients.

(ii) (10 points) **Illustrate its use** to write $f(x,y) = x^2y + x^3 + y^3$ as a polynomial in $(x-1)$ and $(y+1)$ by computing its Taylor expansion about the point $\mathbf{c} = (1,-1)$.

Solution to part (i): Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and suppose that all its partial derivatives of order up through order $N+1$ are continuous on a convex neighborhood D of $\overline{p}_0$. Put

$$\overline{b} = \overline{p} - \overline{p}_0 = (b_1, \ldots, b_n).$$

Then for $\overline{p} \in D$ we have

$$f(\overline{p}) = f(\overline{p}_0) + \sum_{i=1}^{N} \frac{1}{i!} \sum \binom{i}{r_1, r_2, \ldots, r_n} b_1^{r_1} \cdots b_n^{r_n} \frac{\partial^i}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} f(\overline{p}_0)$$
$$+ \frac{1}{(N+1)!} \sum \binom{N+1}{r_1, \ldots, r_n} b_1^{r_1} \cdots b_n^{r_n} \frac{\partial^{N+1}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} f(\overline{w}).$$

Here the sum $\sum \binom{i}{r_1, r_2, \ldots, r_n}$ is over all n-tuples $(r_1, \ldots, r_n)$ such that each $r_j \ge 0$ and $\sum_{j=1}^n r_j = i$. Similarly, the sum $\sum \binom{N+1}{r_1, \ldots, r_n}$ is over all n-tuples $(r_1, \ldots, r_n)$ such that each $r_j \ge 0$ and $\sum_{j=1}^n r_j = N+1$. Finally, $\overline{w}$ is some point on the line segment $L(\overline{p}_0, \overline{p})$.

Solution to part (ii): Let $f : \mathbb{R}^n \to \mathbb{R}$ and all its partial derivatives of order up through order $N+1$ be continuous on a convex neighborhood D of $\mathbf{p}_0$. Then for $\mathbf{p} \in D$,

$$f(\mathbf{p}) = f(\mathbf{p}_0) + \sum_{i=1}^{N} \frac{1}{i!} \left[(\mathbf{p} - \mathbf{p}_0) \circ \triangledown\right]^i f(\mathbf{p}_0) + \frac{1}{(N+1)!} \left[(\mathbf{p} - \mathbf{p}_0) \circ \triangledown\right]^{N+1} f(\mathbf{z})$$

for some point $\mathbf{z}$ on the line segment $L(\mathbf{p}_0, \mathbf{p})$.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ we have seen that there are n^k partial derivatives of order k that can be formed. Continuity of all these partials at a point $\mathbf{c}$ implies that

$D_{i_1,\ldots,i_k} f(\mathbf{c})$ is unchanged when the indices are permuted. So if r_i is the number of times that differentiation with respect to the ith variable occurs, $1 \le i \le n$, $0 \le r_i \le k$, then the multinomial coefficient $\binom{k}{r_1 \cdots r_n} = \frac{k!}{r_1! \cdots r_n!}$ gives the number of permutations of the indices $i_1, \ldots, i_k$.

Write $\beta = \mathbf{p} - \mathbf{p}_0 = (b_1, \ldots, b_n)^T$. Then $\beta \cdot \bigtriangledown = \sum_{i=1}^n b_i \cdot \frac{\partial}{\partial x_i}$. If we know that $b_i \frac{\partial}{\partial x_i} \cdot b_j \frac{\partial}{\partial x_j} = b_j \frac{\partial}{\partial x_j} \cdot b_i \frac{\partial}{\partial x_i}$ for $1 \le i, j \le n$, then

$$(\beta \cdot \bigtriangledown)^k = \left(\sum b_i \frac{\partial}{\partial x_i} \right)^k = \sum \binom{k}{r_1, \ldots, r_n} b_1^{r_1} b_2^{r_2} \cdots b_n^{r_n} \frac{\partial^k}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}. \tag{2}$$

We now apply this to the given function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 y + x^3 + y^3$.

Solution: $f(1, -1) = -1$. $\frac{\partial f}{\partial x}\Big|_{(1,-1)} = (2xy + 3x^2)\big|_{(1,-1)} = 1$. $\frac{\partial f}{\partial y}\Big|_{(1,-1)} = (x^2 + 3y^2)\big|_{(1,-1)} = 4$. $f_{xx}(1, -1) = (2y + 6x)|_{(1,-1)} = 4$. $f_{xy}(1, -1) = 2x|_{(1,-1)} = 2$. $f_{yy}(1, -1) = 6y|_{(1,-1)} = -6$. $f_{xxx}(1, -1) = 6|_{(1,-1)} = 6$. $f_{xxy} = 2$. $f_{yyy} = 6$.

Since all third order partials are constants, all higher order partials are zero. If $\mathbf{p} = (x, y)$, then $\mathbf{p} - \mathbf{c} = (x - 1, y + 1) = \beta$. Then

$$\beta \cdot \bigtriangledown = (x - 1) \frac{\partial}{\partial x} + (y + 1) \frac{\partial}{\partial y}.$$

$$(\beta \cdot \bigtriangledown)^2 = (x - 1)^2 \frac{\partial^2}{\partial x^2} + 2(x - 1)(y + 1) \frac{\partial^2}{\partial x \partial y} + (y + 1)^2 \frac{\partial^2}{\partial y^2}.$$

$$(\beta \cdot \bigtriangledown)^3 = \binom{3}{3,0} (x - 1)^3 \frac{\partial^3}{\partial x^3} + \binom{3}{2,1} (x - 1)^2 (y + 1) \frac{\partial^3}{\partial x^2 \partial y} + \\ + \binom{3}{1,2} (x - 1)(y + 1)^2 \frac{\partial^3}{\partial x \partial y^2} + \binom{3}{0,1} (y + 1)^3 \frac{\partial^3}{\partial y^3}.$$

So

$$f(x, y) = f(1, -1) + \frac{1}{1!} \left[\binom{1}{1,0} (x - 1) \frac{\partial f}{\partial x}(1, -1) + \binom{1}{0,1} (y + 1) \frac{\partial f}{\partial y}(1, -1) \right] + \\ + \frac{1}{2!} \left[\binom{2}{2,0} (x - 1)^2 \frac{\partial^2 f}{\partial x^2}(1, -1) + \binom{2}{1,1} (x - 1)(y + 1) \frac{\partial^2 f}{\partial x \partial y}(1, -1) + \right. \\ \left. + \binom{2}{0,2} (y + 1)^2 \frac{\partial^2 f}{\partial y^2}(1, -1) \right] + \\ + \frac{1}{3!} \left[\binom{3}{3,0} (x - 1)^3 \frac{\partial^3 f}{\partial x^3}(1, -1) + \binom{3}{2,1} (x - 1)^2 (y + 1) \frac{\partial^3 f}{\partial x^2 \partial y}(1, -1) + \right. \\ \left. + \binom{3}{1,2} (x - 1)(y + 1)^2 \frac{\partial^3 f}{\partial x \partial y^2}(1, -1) + \binom{3}{0,3} (y + 1)^3 \frac{\partial^4 f}{\partial y^3}(1, -1) \right] + 0.$$

So finally

$$f(x, y) = -1 + [(x - 1) + 4(y + 1)] + \left[2(x - 1)^2 + 2(x - 1)(y + 1) - 3(y + 1)^2 \right] + \\ + \left[(x - 1)^3 + (x - 1)^2 (y + 1) + (y + 1)^3 \right].$$

6. Define a function $\overline{F}\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\overline{F}\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1x_3 + 2x_2x_3 + 4 \\ x_1^2x_3 + x_1x_2 + 2 \end{pmatrix}.$$

(i) (4 points) It should be clear that $\overline{F}$ is differentiable on all of $\mathbb{R}^3$. Compute the Jacobian $J_{\overline{F}}$.

(ii) (4 points) Let $\overline{p}_0 = (2, -3, 1)$. Check to see that $\overline{F}(\overline{p}_0) = \overline{0}$ and evaluate $J_{\overline{F}}(\overline{p}_0)$.

(iii) (4 points) Name the theorem (Call it Theorem A) that guarantees that some pairs of variables from $\{x_1, x_2, x_3\}$ can be written as functions of the third variable. List these pairs.

(iv) (8 points) Choose one of the pairs you selected in part (iii). If i, j, k are 1, 2, 3 in some order, suppose that you pick x_i and x_j to write as a function of x_k. Then Theorem A says that there is a differentiable function $\overline{G}$ that gives x_i and x_j as functions of x_k. Compute the Jacobian matrix for $\overline{G}$ at the special value of x_k determined by the original point $\overline{p}_0$. Now do this for all the other special pair(s) you selected in part (iii).

Solution Part (i).

$$J_{\overline{F}} = \begin{pmatrix} x_3 & 2x_3 & x_1 + 2x_2 \\ 2x_1x_3 + x_2 & x_1 & x_1^2 \end{pmatrix}.$$

Part (ii). It is easy to check that $\overline{F}(\overline{p}_0) = \overline{0}$. Then

$$J_{\overline{F}}(\overline{p}_0) = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 4 \end{pmatrix}.$$

Part (iii). Theorem A is the Implicit Function Theorem It is clear that the first two columns are linearly dependent, but the first and third columns are independent and the second and third columns are independent. So the appropriate pairs are $\{x_1, x_3\}$ and $\{x_2, x_3\}$.

Part (iv). We first choose x_1 and x_3 to write as a function of x_2. Reorder the variables as (x_1, x_3, x_2). Then the appropriate Jacobian at the point $\overline{p_0}' = (2, 1, -3)$ is

$$J_{\overline{F}}(\overline{p}_0) = \begin{pmatrix} 1 & -4 & 2 \\ 1 & 4 & 2 \end{pmatrix}.$$

Then $\overline{F}(\overline{G}(x_2), x_2) = 0$ in a neighborhood of $x_2 = -3$ and

$$J_{\overline{G}}(-3) = -\begin{pmatrix} 1 & -4 \\ 1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{-1}{8} \begin{pmatrix} 4 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 16 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Secondly, we choose x_2 and x_3 to write as a function of x_1. Reorder the variables as (x_2, x_3, x_1). Then the appropriate Jacobian at the point $\overline{p_0}' = (-3, 1, 2)$ is

$$J_{\overline{F}}(\overline{p}_0') = \begin{pmatrix} 2 & -4 & 1 \\ 2 & 4 & 1 \end{pmatrix}.$$

Then $\overline{F}(\overline{G}(x_1), x_1) = 0$ in a neighborhood of $x_1 = 2$ and

$$J_{\overline{G}}(2) = -\begin{pmatrix} 2 & -4 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{-1}{16}\begin{pmatrix} 4 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}.$$

7. This problem has three parts.

   (i) (5 points) Suppose that f is continuous at x_0 and $f'(x)$ exists in a deleted neighborhood of x_0 and $\lim_{x\to x_0} f'(x) = L \in \mathbb{R}$. Show that $f'(x_0)$ exists and $f'(x)$ is continuous at x_0.

   Solution: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$, if this limit exists in $\mathbb{R}$.
   Let $L(x_0, x_0+h)$ denote the open interval between x_0 and x_0+h, where h could be either positive or negative. Similarly, let $L[(x_0, x_0+h]$ be the closed interval between x_0 and x_0+h. Then f is continuous on $L[x_0, x_0+h]$ for small h and differentiable on $L(x_0, x_0+h)$. Hence by the MVT there is a $c_h \in L(x_0, x_0+h)$ such that $\frac{f(x_0+h)-f(x_0)}{h} = f'(c_h)$. As $h \to 0$ clearly $c_h \to x_0$, so

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} f'(c_h) = \lim_{c_h\to x_0} f'(c_h) = L.$$

   Hence $f'(x_0) = L$ and $\lim_{x\to x_0} f'(x) = f'(x_0)$ so $f'(x)$ is continuous at x_0.

   (ii) (5 points) For each integer $n \geq 0$ show that $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$.

   Solution: Clearly if $n = 0$, $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$. So suppose that $n \geq 1$.

$$\begin{aligned} \left|\frac{e^{-\frac{1}{x^2}}|}{x^n}\right| &\leq |x^n| \\ \iff e^{-\frac{1}{x^2}} &\leq (x^n)^2 \\ \iff -\frac{1}{x^2} \leq n\log(x^2) = -n\log\left(\frac{1}{x^2}\right) \\ \iff \frac{1}{x^2} &\geq n \cdot \log\left(\frac{1}{x^2}\right). \end{aligned}$$

Using L'Hospital's Rule it is easy to check that $\lim_{y\to\infty} \frac{y}{\log(y)} = +\infty$. So with $y = \frac{1}{x^2}$ above we see that for $|x|$ small enough we have

$$\frac{1}{x^2} \geq n \cdot \log\left(\frac{1}{x^2}\right).$$

Hence

$$0 \leq \lim_{x\to 0} \left|\frac{e^{-\frac{1}{x^2}}|}{x^n}\right| \leq \lim_{x\to 0} |x^n| = 0.$$

(iii) (10 points) Define

$$f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is infinitely differentiable on $\mathbb{R}$ and that $f^{(n)}(0) = 0$ for all $n \geq 0$. Use this to show that there is no power series $\sum_{n=0}^{\infty} a_n x^n$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all x in some interval $(-r, r)$ for $r > 0$.

Solution: Since $\lim_{x\to 0} f(x) = 0$ we see that f is continuous at $x = 0$. For $x \neq 0$, $f'(x) = e^{-\frac{1}{x^2}} 2x^{-3}$. Then $\lim_{x\to 0} f'(x) = 0$ by Part (ii) and $f'(0)$ exists and equals 0 by Part (i). So $f'(x)$ is continuous at 0.

For $x \neq 0$, $f^{(2)}(x) = e^{-\frac{1}{x^2}}[4x^{(-3)2} - 6x^{(-3)2+2}]$ (after a little computation). Since $f'(x)$ is continuous at 0, $f^{(2)}(x)$ exists for $x \neq 0$ and

$$\lim_{x\to 0} f^{(2)}(x) = \lim_{x\to 0} \left[\frac{4e^{-\frac{1}{x^2}}}{x^6} - \frac{6e^{-\frac{1}{x^2}}}{x^4}\right] = 0$$

by Part (ii). So $f^{(2)}(0) = 0$ and $f^{(2)}(x)$ is continuous everywhere on $\mathbb{R}$. We want to see what is the general form of $f^{(n)}(x)$ for $x \neq 0$. Suppose that we have shown that for $x \neq 0$

$$f^{(n)}(x) = e^{-\frac{1}{x^2}} P_n(x^{-1})$$

where $P_n(x^{-1})$ is a polynomial in x^{-1} where the degree of each monomial in P_n has the same parity as n, the term of largest degree has degree equal to $3n$, and the monomial of least degree has degree $n + 2$. Also, $f^{(n)}(x)$ is continuous at 0. Then for $x \neq 0$ we have

$$\begin{aligned} f^{(n+1)}(x) &= e^{-\frac{1}{x^2}} \left[2x^{-3} P_n(x^{-1}) + \left(p_n(x^{-1})\right)'\right] \\ &= e^{-\frac{1}{x^2}} P_{n+1}(x). \end{aligned}$$

It is easy to check that P_{n+1} has degree $3(n+1)$ and the monomial of smallest degree has degree equal to $n + 3 = (n+1) + 2$. Each monomial of $P_{n+1}(x^{-1})$

times $e^{-\frac{1}{x^2}}$ has limit 0 as $x \to 0$ by Part (ii). Hence $\lim_{x\to 0} f^{(n+1)}(x) = 0$ So by Part (i), $f^{(n+1)}(0) = 0$ and $f^{(n+1)}(x)$ is continuous on $\mathbb{R}$. By the Principle of Mathematical Induction, this implies that $f^{(n)}(0) = 0$ for all $n \geq 0$.

If $f(x)$ were represented by a power series (of the form given above) in some interval $(-r, r)$ with $r > 0$, it would have to be the Maclaurin series (i.e., the Taylor series about the point 0). This is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}, \text{ i.e., the identically zero series.}$$

But $f(x)$ is not identically zero on any open neighborhood containing 0, so it is not given by a Maclaurin series there.

8. Prove the following:

   (i) (6 points) If $p > 1$ and $0 < x < 1$, then $1 - px < (1-x)^p$.

   (ii) (6 points) Let $p > 1$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of **positive** numbers for which there is an n_0 such that $a_{n+1}/a_n \leq 1 - p/n$ for all $n \geq n_0$. Then $\sum_{n=1}^{\infty} a_n$ converges. (We will call this the S Test.)

   (iii) (8 points) Use the S Test to determine for which $a > 0$, $b > 0$ the following (hypergeometric) series converges:

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)b+2)} + \cdots$$

Solution: (i) Taylor's formula for $f(x) = (1-x)^p$ expanded about 0 gives

$$f(x) = f(0) + f'(0)x + f''(\zeta)\frac{x^2}{2}, \text{ for some } \zeta \in (0, x),$$

since f and f' are continuous on $[0, x]$ and f'' exists on $(0, x)$. This implies

$$(1-x)^p = 1 - px + p(p-1)\frac{(1-\zeta)^{p-2}}{2} > 1 - px,$$

since $p > 1$ and $0 < \zeta < x < 1$.

(ii) Compare the given series with $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which is convergent since $p > 1$.

Put $c = a_{n_0} n_0^p$ where n_0 is the integer given in the hypothesis. Then note that

$$a_{n_0} \leq \frac{a_{n_0} n_0^p}{n_0^p} = \frac{c}{n_0^p}.$$

For each $n \geq n_0$ let $P(n)$ be the statement that $a_n \leq \frac{c}{n^p}$. We have just observed that $P(n_0)$ is true. We will prove by induction that $P(n)$ is true for all $n \geq n_0$. So suppose that for some $n \geq n_0$ we have that $P(n)$ is true and try to show that $P(n+1)$ is true.

So suppose that for some $n \geq n_0$ we have $a_n \leq \frac{c}{n^p}$. Then

$$\begin{aligned} a_{n+1} &= \frac{a_{n+1}}{a_n} a_n \leq \left(1 - \frac{p}{n}\right) a_n \\ &< \left(1 - \frac{1}{n}\right)^p \frac{c}{n^p} \quad \text{(by part (i) with } x = \frac{1}{n}) \\ &= c\frac{(n-1)^p}{n^{2p}} = \frac{c}{(n+1)^p} \frac{(n+1)^p(n-1)^p}{n^{2p}} \\ &= \frac{c}{(n+1)^p} \frac{(n^2-1)^p}{(n^2)^p} < \frac{c}{(n+1)^p}. \end{aligned}$$

Hence $P(n+1)$ is true, so $P(n)$ is true for all $n \geq n_0$. Thus it follows that $\sum a_n$ converges since $\sum \frac{c}{n^p}$ converges.

(iii) Put $a_1 = \frac{a}{b}$, and in general

$$a_n = \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{a+n}{b+n} = 1 - \frac{b-a}{b+n} = 1 - \frac{\frac{b-a}{1+\frac{b}{n}}}{n}.$$

First suppose that $b - a > 1$. Then $\lim_{n\to\infty} \frac{b-a}{1+\frac{b}{n}} = b - a > 1$. Choose a p with $1 < p < b - a$. Then there is an n_0 such that $n \geq n_0 \implies \frac{b-a}{1+\frac{b}{n}} > p$, so that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{b-a}{1+\frac{b}{n}} < 1 - \frac{p}{n}.$$

Then by Raabe's Test we see that $\sum a_n$ converges.

If $b - a = 1$, the series becomes

$$\frac{a}{a+1} + \frac{a(a+1)}{(a+1)(a+2)} + \frac{a(a+1)(a+2)}{(a+1)(a+2)(a+3)} + \cdots$$

$$= \frac{a}{a+1} + \frac{a}{a+2} + \frac{a}{a+3} + \cdots$$

which is divergent by comparison with $\sum \frac{1}{n}$. In detail:

$$\frac{a}{a+n} = \frac{1}{n}\frac{a}{1+\frac{a}{n}} > \frac{1}{n}\frac{a}{2} \text{ for } n > a.$$

At this point we know that the hypergeometric series converges if $b - a > 1$ and diverges if $b - a = 1$. Suppose that $b - a < 1$, i.e., $0 < b < a + 1$. Look at the terms in the series. As b gets smaller, the fractions that are the terms get larger, so since the series diverges for $b = a + 1$, it certainly diverges for $b < a + 1$.