University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
January 11, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Julien Langou, Weldon Lodwick, Jan Mandel (Chair)

1. Prove that every sequence of real numbers contains a monotone subsequence.

2. Prove or disprove that if M is infinite compact subset of $\mathbb{R}$, then M contains a nondegenerate interval. (A nondegenerate interval is off the form $(a, b]$, $[a, b]$, (a, b), $[a, b)$ with $a < b$.)

3. Show that in a neighborhood of $(0,0,0,0)$ the system of equations

$$\begin{aligned} 3w + x - y + z^2 &= 0 \\ w - x + 2y + z &= 0 \\ 2w + 2x - 3y + 2z &= 0 \end{aligned}$$

can be solved for w, x, z in terms of y; for w, y, z in terms of x; for x, y, z in terms of w; but not for w, x, y in terms of z.

4. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ defined by

$$\forall\, y \in \mathbb{R}, \quad g(y) = \int_0^1 f(x) e^{xy} dx.$$

(a) Show that g is continuous.

(b) Show that $\lim_{y \to -\infty} g(y) = 0$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists α in $\mathbb{R}$ and β in $\mathbb{R}$ such that $\lim_{x \to -\infty} f(x) = \alpha$ and $\lim_{x \to +\infty} f(x) = \beta$. Show that f is uniformly continuous on $\mathbb{R}$, and bounded.

6. Let $X \subset \mathbb{R}$, and $(f_n : X \to \mathbb{R})_{n \in \mathbb{N}}$ be a sequence of functions uniformly continuous on X and uniformly converging on X to the function $f : X \to \mathbb{R}$. Show that f is uniformly continuous on X.

7. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^n$ when

(a) $c_n = \ln\left(1 + \frac{1}{n}\right)$

(b) c_n is the n-th decimal digit of π.

8. Let E_n be subsets of a metric space and $E = \bigcup_{n=1}^{N} E_n$. Prove that $E' = \bigcup_{n=1}^{N} E'_n$, where A' denotes the set of all limit points of A.