University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
January 11, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Julien Langou, Weldon Lodwick, Jan Mandel (Chair)

1. Prove that every sequence of real numbers contains a monotone subsequence.

   Solution. Let $\{x_n\}$ be a sequence and define $I = \{m | \forall k : k > m \Rightarrow x_m < x_k\}$. Either I is infinite or finite. If it is infinite, we are done since $\{x_m, m \in I\}$ is increasing sequence. If I is finite, then $\exists M \in \mathbb{N}$ such that $n \geq M \Rightarrow n \notin I$. Let $m_1 = M$. Suppose we now have $m_1 < m_2 < ... < m_j$ and $x_{m_1} \geq x_{m_2} \geq ... \geq x_{m_j}$. Since $m_j \notin I$, there exists $k > m_j$ such that $x_{m_j} \geq x_k$, and we choose $m_{j+1} = k$. The sequence $\{x_{m_j}\}$ is the desired monotonic sequence.

2. Prove or disprove that if M is infinite compact subset of $\mathbb{R}$, then M contains a nondegenerate interval. (A nondegenerate interval is off the form $(a,b]$, $[a,b]$, (a,b), $[a,b)$ with $a<b$.)

   Solution. The statement is false. We provide a counterexcmple. Consider $M = \{0\} \cup \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$. The set M contains its only limit point 0, so it is closed. Since $M \subset [0,1]$, it is also bounded and therefore compact. But for any $x \in M$, any interval $(x-\varepsilon, x+\varepsilon)$, $\varepsilon > 0$, contains points not in M so M has empty interior and thus does not cointain any open interval and so any nondegenerate interval.

3. Show that in a neighborhood of $(0,0,0,0)$ the system of equations

$$\begin{aligned} 3w + x - y + z^2 &= 0 \\ w - x + 2y + z &= 0 \\ 2w + 2x - 3y + 2z &= 0 \end{aligned}$$

can be solved for w, x, z in terms of y; for w, y, z in terms of x; for x, y, z in terms of w; but not for w, x, y in terms of z.

Solution. Write the system as $F(w,x,y,z) = 0$. We have $F(0,0,0,0) = 0$, and all partial derivatives of F exist and are continuous at $(0,0,0,0)$. *Note: it is important to state these assumption of the implicit function theorem.* The Jacobian of F at $(0,0,0,0)$ is

$$\begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}.$$

To solve the system for w, x, z, compute

$$\frac{\partial F}{\partial(w,x,z)} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

The determinant of this matrix is $-12 \neq 0$, so the statement follows from the implicit function theorem.

To solve the system for w, y, z, compute

$$\frac{\partial F}{\partial(w,y,z)} = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}.$$

The determinant of this matrix is $21 \neq 0$, so the statement follows from the implicit function theorem.

To solve the system for x, y, z, compute

$$\frac{\partial F}{\partial(w,y,z)} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}.$$

The determinant of this matrix is $3 \neq 0$, so the statement follows from the implicit function theorem.

To solve the system for w, x, y, compute

$$\frac{\partial F}{\partial(w,x,y)} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

The determinant of this matrix is 0, thus the implicit function theorem cannot be applied. *Note that this does not necessarily mean that the system cannot be solved. It only means that it cannot be solved by the implicit function theorem. Since solution in some other way might be possible, further analysis is necessary.* The system for w, x, y is linear with the matrix $\partial F/\partial(w, x, y)$ above, which is singular. Hence, it has either no solution or infinitely many solutions. The system of linear equations

$$\begin{aligned} 3w + x - y &= -z^2 \\ w - x + 2y &= -z \\ 2w + 2x - 3y &= -2z \end{aligned}$$

is equivallent to

$$\begin{aligned} 4w + y &= -z \\ 4x - 7y &= 0 \\ 0 &= 3z - z^2 \end{aligned}$$

(We did $R1 \leftrightarrow R2$, $R2 \leftarrow R2 - 2R1$, $R3 \leftarrow R3 - 3R1$, $R3 \leftarrow R3 - R2$, $R1 \leftarrow R1 + R2$.) The last equation fixes z (to be either 0 or 3) and therefore it is not possible the write w, x, y as function of z.

4. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ defined by

$$\forall\, y \in \mathbb{R}, \quad g(y) = \int_0^1 f(x) e^{xy} dx.$$

(a) Show that g is continuous.

(b) Show that $\lim_{y \to -\infty} g(y) = 0$.

Solution.

(a) Let $y \in \mathbb{R}$. $\forall h \in \mathbb{R}$,

$$\begin{aligned} |g(y+h) - g(y)| &= |\int_0^1 f(x) e^{xy} (e^{xh} - 1) dx|, \\ &\leq \int_0^1 |f(x)| e^{xy} |e^{xh} - 1| dx, \\ &\leq (e^{|h|} - 1) \int_0^1 |f(x)| e^{xy} dx. \end{aligned}$$

But

$$\lim_{h \to 0} (e^{|h|} - 1) = 0$$

so,

$$\lim_{h \to 0} |g(y+h) - g(y)| = 0.$$

Thus, g is continuous on $\mathbb{R}$.

(b) Denote $M = \sup_{x \in [0;1]} |f(x)|$, we have:

$$\forall y \in \mathbb{R}, \quad |g(y)| \leq M \int_0^1 e^{xy} dx = M \frac{e^y - 1}{y}.$$

But

$$\lim_{y \to -\infty} \frac{e^y - 1}{y} = 0,$$

so,

$$\lim_{y \to -\infty} g(y) = 0.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists α in $\mathbb{R}$ and β in $\mathbb{R}$ such that $\lim_{x\to-\infty} f(x) = \alpha$ and $\lim_{x\to+\infty} f(x) = \beta$. Show that f is uniformly continuous on $\mathbb{R}$, and bounded.

   Solution. Let $\varepsilon > 0$. Since $\lim_{x\to-\infty} f(x) = \alpha$, there exists M such that $|f(x) - \alpha| < \varepsilon/2$ if $x < M$. Similarly, since $\lim_{x\to+\infty} f(x) = \beta$, there exists N such that $|f(x) - \beta| < \varepsilon/2$ if $x > N$. The function f is continuous on the interval $[M-1, N+1]$, thus uniformly continuous, so there exists $\delta > 0$ such that if $x, y \in [M-1, N+1]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, and without loss of generality, $\delta < 1$. Now let $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $x < y$. Since $x < y$ and $y - x < 1$, one or more of the following holds true: $x < y < M$, $M - 1 < x < y < N + 1$, or $N < x < y$. If $x < y < M$, then $|f(x) - f(y)| \le |f(x) - \alpha| + |f(y) - \alpha| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. If $M - 1 < x < y < N + 1$, then $|f(x) - f(y)| < \varepsilon$ by the uniform continuity. If $N < x < y$, then $|f(x) - f(y)| \le |f(x) - \beta| + |f(y) - \beta| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. In any case, $|f(x) - f(y)| < \varepsilon$.

   To prove that f is bounded, choose $\varepsilon = 1$ above. Since f is continuous on the closed and bounded interval $[M, N]$, there exists C such that $|f(x)| < C$ for all $x \in [M, N]$, and so $|f(x)| < \max\{C, |\alpha| + 1/2, |\beta| + 1/2\}$ for all $x \in \mathbb{R}$.

6. Let $X \subset \mathbb{R}$, and $(f_n : X \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of functions uniformly continuous on X and uniformly converging on X to the function $f : X \to \mathbb{R}$. Show that f is uniformly continuous on X.

   Solution. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, we have

$$\exists N \forall n \geq N \forall x \in X : |f(x) - f_n(x)| < \varepsilon/3.$$

   In particular,

$$\forall x \in X : |f(x) - f_N(x)| < \varepsilon/3.$$

   Since $f_N : X \to \mathbb{R}$ is uniformly continuous on X,

$$\exists \eta > 0 \forall x', x'' \in X : |x' - x''| < \eta \Longrightarrow |f_N(x') - f_N(x'')| < \varepsilon/3.$$

   Let x',$x'' \in \mathbb{R}$ such that $|x' - x''| < \eta$. Then, by the triangle inequality,

$$\begin{aligned}
|f(x') - f(x'')| &= |f(x') - f_N(x') + f_N(x') - f_N(x'') + f_N(x'') - f(x'')| \\
&\leq |f(x') - f_N(x')| + |f_N(x') - f_N(x'')| + |f(x'') - f_N(x'')| \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

   This proves that f is uniformly continuous.

7. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^n$ when

(a) $c_n = \ln\left(1 + \frac{1}{n}\right)$

(b) c_n is the n-th decimal digit of π.

Solution.

(a) $\frac{c_{n+1}}{c_n} = \frac{\ln\left(1+\frac{1}{n+1}\right)}{\ln\left(1+\frac{1}{n}\right)} = \frac{\frac{1}{n+1}+o\left(\frac{1}{n}\right)}{\frac{1}{n}+o\left(\frac{1}{n}\right)} \to 1$ as $n \to \infty$, so using d'Alembert rule, $R = 1$.

(b) Since $|c_n x^n| \leq 9|x|^n$ and $\sum_{n=1}^{\infty} 9x^n$ has a radius of convergence of 1, we have $R \geq 1$. We now study the power series $\sum_{n=1}^{\infty} c_n x^n$ at $x = 1$. We have $c_n \geq 0$ for all n, and $c_n \geq 1$ for infinitely many n since π is irrational, so $\sum_{n=1}^{\infty} c_n$ diverges. Thus, $R \leq 1$. In conclusion, $R = 1$.

8. Let E_n be subsets of a metric space and $E = \bigcup_{n=1}^{N} E_n$. Prove that $E' = \bigcup_{n=1}^{N} E'_n$, where A' denotes the set of all limit points of A.

Solution. Recall that x is a limit point of S if and only if $\forall \varepsilon > 0 \exists y \in S : 0 < d(x, y) < \varepsilon$.

Let $x \in \bigcup_{n=1}^{N} E'_n$. Then $x \in E'_n$ for some n, and $x \in E'$ because $E_n \subset E$.

Let $x \in E'$. Then $\forall k \in \mathbb{N} \exists y_k \in E : 0 < d(x, y_k) < 1/k$. Since each y_k is in one of the finitely many sets E_n, there exists E_n which contains infinitely many y_k. We show that $x \in E'_n$. Let $\varepsilon > 0$. Since E_n contains infinitely many points y_k, it contains at least one point y_k such that $1/k < \varepsilon$. Then $0 < d(x, y_k) < 1/k < \varepsilon$. Consequently, $x \in \bigcup_{n=1}^{N} E'_n$.