University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
June 5, 2009

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Andrew Knyazev, Julien Langou, Jan Mandel (Chair)

1. Let $E \subset \mathbb{R}$ be nonempty and bounded above. Prove that $\sup \overline{E} = \sup E$.

2. Construct a function such that $\lim_{c\to0^+}\int_c^1 f(x)\,dx$ exists, but $\lim_{c\to0^+}\int_c^1 |f(x)|\,dx$ does not. Justify the answer.

3. Define $f(0,0) = 0$ and $f(x_1, x_2) = \frac{x_1^3}{x_1^2 + x_2^2}$ for $(x_1, x_2) \neq (0,0)$. Prove that the directional derivatives of f exist at all points of $\mathbb{R}^2$, but $f'(0,0)$ does not exist.

4. If E is nonempty subset of a metric space X with distance function d, define the distance of $x \in X$ to E by $\rho_E(x) = \inf_{y \in E} d(x, y)$. Prove that ρ_E is uniformly continuous on X.

5. Let $\mathcal{L}^1$ denotes the linear space of real sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n|$ converges. We equip $\mathcal{L}^1$ with the norm $\|x\|_{\mathcal{L}^1} = \sum_{n=1}^{\infty} |x_n|$, where $x = (x_1, x_2, \ldots)$. Construct a *countable* subset $S \subset \mathcal{L}^1$, which is dense in $\mathcal{L}^1$, i.e., $\bar{S} = \mathcal{L}^1$, where the closure is taken with respect to the given norm $\| \cdot \|_{\mathcal{L}^1}$. Prove that the set S has the desired properties.

6. Compute $\sum_{n=0}^{\infty}(n+1)x^n$.

7. Let E_n be subsets of a metric space and $E = \bigcup_{n=1}^{N} E_n$. For a given set S we denote the set of its limits (or accumulation) points by $Lim(S)$. Prove that $Lim(E) = \bigcup_{n=1}^{N} Lim(E_n)$.

8. Prove that if $|x| < 1$ and the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.