University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
June 5, 2009

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Andrew Knyazev, Julien Langou, Jan Mandel (Chair)

1. Let $E \subset \mathbb{R}$ be nonempty and bounded above. Prove that $\sup \overline{E} = \sup E$.

Solution

Since E is nonempty and bounded above, $\sup E$ exists as a real number. We will show that $\sup \overline{E}$ exists. Since E is nonempty, so is $\overline{E}$.Since E is bounded above, so is $\overline{E}$. Otherwise, if M is upper bound on E and $\overline{E}$ is not bounded above, there exists $y \in \overline{E}$ such that $y > M + 1$; but there is no $x \in E \cap (M + 1/2, M + 3/3)$, contradiction. Thus $\sup \overline{E}$ exists as a real number.

By property of the closure, since $E \subset \overline{E}$, we have

$$\sup E \leq \sup \overline{E}.$$

It remains to prove that $\sup \overline{E} \leq \sup E$. Let $\varepsilon > 0$. By the definition of supremum, $\exists\ x \in \overline{E}$ such that

$$x \geq \sup \overline{E} - \varepsilon/2.$$

(Note that it was necessary to show that $\sup \overline{E}$ is real number otherwise we could not subtract.) Otherwise we would have that $\forall\ x \in \overline{E} : x < \sup \overline{E} - \varepsilon/2$ which makes $\sup \overline{E} - \varepsilon/2$ a upper bound for $\overline{E}$ smaller than $\sup \overline{E}$; and that would contradict the definition of $\sup \overline{E}$. From the definition of closure,

$$\exists y \in E : x - \varepsilon/2 \leq y \leq x + \varepsilon/2,$$

thus

$$\exists y \in E : y \geq \sup \overline{E} - \varepsilon.$$

Consequently, $\sup E \geq \sup \overline{E} - \varepsilon$ for any $\varepsilon > 0$, $\sup E \geq \sup \overline{E}$.

2. Construct a function such that $\lim_{c\to 0^+}\int_c^1 f(x)\,dx$ exists, but $\lim_{c\to 0^+}\int_c^1 |f(x)|\,dx$ does not. Justify the answer.

Solution

We will construct a function f such that the areas between the graph and the x axis form a nonabsolutely convergent series. Then we need to show that the values of the integral are close to a partial sums of the series.

Define

$$f(x)=\begin{cases} n \text{ if } \frac{1}{N+1}<x\le\frac{1}{N},\ N \text{ even} \\ -n \text{ if } \frac{1}{N+1}<x\le\frac{1}{N},\ N \text{ odd}\end{cases}$$

Let $c>0$ and define $N=\left\lfloor\frac{1}{c}\right\rfloor$. Then

$$\frac{1}{N+1}<c\le\frac{1}{N},$$

and

$$\int_c^1 |f(x)|dx \ge \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)(n)=\sum_{n=1}^{N}\frac{1}{n+1}.$$

When $c\to 0^+$, $N\to\infty$, and we see that

$$\int_c^1 |f(x)|dx\to+\infty$$

because the harmonic series diverges.

On the other hand,

$$\begin{aligned}\int_c^1 f(x)dx &= 1\left(1-\frac{1}{2}\right)+\cdots+(-1)^{N-1}(N)\left(\frac{1}{N-1}-\frac{1}{N}\right)+\int_c^{1/N} f(x)dx \\ &= \frac{1}{2}-\frac{1}{3}+\cdots+\frac{(-1)^{N-1}}{N-1}+r_N,\quad |r_N|\le\frac{1}{N}.\end{aligned}$$

When $c\to 0$, $N\to\infty$, and we see that

$$\int_c^1 f(x)dx\to\sum_{n=2}^{\infty}\frac{(-1)^n}{n}$$

which is convergent by the alternating series test.

3. Define $f(0,0) = 0$ and $f(x_1, x_2) = \frac{x_1^3}{x_1^2 + x_2^2}$ for $(x_1, x_2) \neq (0,0)$. Prove that the directional derivatives of f exist at all points of $\mathbb{R}^2$, but $f'(0,0)$ does not exist.

Solution

The directional derivatives of f exist at all points of $\mathbb{R}^2 \setminus \{(0,0)\}$ since f is a composition of functions who all have well defined directional derivatives for any points in $\mathbb{R}^2 \setminus \{(0,0)\}$. It remains to consider the point $(0,0)$.

By the definition of the directional derivative $(D_u f)(x)$ in the direction $u = (u_1, u_2)$,

$$(D_u f)(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t \to 0} \left(\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} \right) \frac{1}{t} = \frac{u_1^3}{u_1^2 + u_2^2}.$$

So, the directional derivative $(D_u f)(x)$ exists at the point $x = (0,0)$ for any vector u. However, $(D_u f)(0,0)$ is not a linear function of the variables u_1 and u_2, hence $f'(0,0)$ does not exist.

Remark: You will find more possible (and interesting!) questions related to this function reading problem 9.14 of Rudin (page 240).

4. If E is nonempty subset of a metric space X with distance function d, define the distance of $x \in X$ to E by $\rho_E(x) = \inf_{y \in E} d(x, y)$. Prove that ρ_E is uniformly continuous on X.

Solution

Take any $p, q \in X$ and $x \in E$. From the triangle inequality,

$$d(p, x) \leq d(p, q) + d(q, x).$$

Taking infimum, we have

$$\rho_E(p) \leq d(p, q) + \rho_E(q),$$

thus

$$\rho_E(p) - \rho_E(q) \leq d(p, q),$$

and switching p and q and using symmetry of d,

$$\rho_E(q) - \rho_E(p) \leq d(p, q),$$

thus

$$|\rho_E(p) - \rho_E(q)| \leq d(p, q).$$

Therefore, $\forall\, \varepsilon > 0$, $\exists\, \delta$ (for example $\delta = \varepsilon$) such that, $\forall\, p$ and $q \in E$ such that $d(p, q) \leq \delta$, we have

$$|\rho_E(p) - \rho_E(q)| \leq \varepsilon.$$

This is the definition of the uniform continuity of the function ρ_E as a mapping of the metric space X with distance function d into the metric space $\mathbb{R}$ with distance function $|x - y|$.

5. Let $\mathcal{L}^1$ denotes the linear space of real sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n|$ converges. We equip $\mathcal{L}^1$ with the norm $\|x\|_{\mathcal{L}^1} = \sum_{n=1}^{\infty} |x_n|$, where $x = (x_1, x_2, \ldots)$. Construct a *countable* subset $S \subset \mathcal{L}^1$, which is dense in $\mathcal{L}^1$, i.e., $\bar{S} = \mathcal{L}^1$, where the closure is taken with respect to the given norm $\| \cdot \|_{\mathcal{L}^1}$. Prove that the set S has the desired properties.

Solution

Let S be a set of sequences with all rational coefficients, the finite number of which are non-zero. $S \subset \mathcal{L}^1$ since the number of non-zeros is finite for any sequence in S.

First, we show that S is countable. Since the set of all rationals $\mathcal{Q}$ is countable, for each N, the set

$$S_N = \{(x_n) : x_n \in \mathbb{Q}, x_n = 0 \forall n > N\}$$

is countable because a finite product of countable sets is countable. The set S is countable as the countable union countable sets,

$$S = \bigcup_{N \in \mathbb{N}} S_N.$$

Now we show that $\bar{S} = \mathcal{L}^1$. We take arbitrary $\epsilon > 0$ and $x \in \mathcal{L}^1$, and we need to show that x can be approximated in the $\| \cdot \|_{\mathcal{L}^1}$ norm by elements of S with ϵ accuracy. Since $\|x\|_{\mathcal{L}^1} = \sum_{n=1}^{\infty} |x_n| < \infty$, by the series convergence definition $\exists N$ such that $\sum_{n=N+1}^{\infty} |x_n| < \epsilon/2$. Let y be a sequence that retains the first N coefficients of the sequence x and has a tail of zeros, then $\|x - y\|_{\mathcal{L}^1} = \sum_{n=N+1}^{\infty} |x_n| < \epsilon/2$. Now, we take each coefficient y_i, $i = 1, \ldots, N$ and approximate it by a rational number z_i, $i = 1, \ldots, N$ such that $|y_i - z_i| < \epsilon/(2N)$ using that fact that the set of rational numbers is dense on the real line with respect to the standard norm of a real number given by its absolute value. Finally, we take $z = (z_1, \ldots, z_N, 0, 0, \ldots)$ and conclude that $z \in S$ and $\|x - z\|_{\mathcal{L}^1} < \epsilon$.

6. Compute $\sum_{n=0}^{\infty}(n+1)x^n$.

Solution

Let us denote $a_n = n+1$ and $v_n(x) = (n+1)x^n = a_n x^n$. First, we need to analyze the series convergence. If $|x| \geq 1$ the series diverges since the necessary convergence criteria, $v_n(x) \to 0$, is violated. If $|x| < 1$ the series converges as a power series at the origin with the convergence radius

$$R = 1/\lim_{n\to+\infty} |a_{n+1}|/|a_n| = 1/\lim_{n\to+\infty} \frac{n+2}{n+1} = 1.$$

Moreover, for any $r \in (0,1)$ the power series converges uniformly in $|x| < r$.

Let us denote $u_n(x) = x^n$, and notice that

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$ as a sum of a geometric progression.

We observe that $(u_0(x))' = 0$ and $(u_n(x))' = nx^{n-1} = v_{n-1}(x)$ for $n > 0$. Both series $\sum_{n=0}^{\infty} v_n(x)$ and $\sum_{n=0}^{\infty} u_n(x)$ have the same radius of convergence $R = 1$. Thus,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} u_n(x)\right)' = \sum_{n=0}^{\infty} v_n(x) = \sum_{n=0}^{\infty}(n+1)x^n \quad \forall\, |x| < 1.$$

by the theorem that power series can be differentiated term by term within the circle of convergence. Alternatively, noting that for any $r < 1$, $\sum_{n=0}^{\infty} u_n'(x)$ converges uniformly for $|x| < r$, and $\sum_{n=0}^{\infty} u_n(0)$ converges. Given $|x| < 1$ chose $r = (|x|+1)/2$ to get that the equality above holds for that x.

7. Let E_n be subsets of a metric space and $E = \bigcup_{n=1}^{N} E_n$. For a given set S we denote the set of its limits (or accumulation) points by $Lim(S)$. Prove that $Lim(E) = \bigcup_{n=1}^{N} Lim(E_n)$.

Solution

In a metric space with a distance $d(\cdot,\cdot)$, a point x is called a limit point of a given set S if for every $\epsilon > 0$ the open ball defined by

$$B_\epsilon(x) = \{y : d(x,y) < \epsilon\}$$

contains a point of S other than x itself. Since

$$\bigcup_{n=1}^{N} (E_n) = (((E_1 \cup E_2) \cup E_3) \cdots \cup E_N)$$

we can take $N = 2$ without loosing generality to simplify the notation.

By definition, $x \in Lim(E_i)$, $i = 1$ or 2 means that $\forall \epsilon > 0 \quad \exists z \in E_i$ such that $d(x,z) < \epsilon$ and $z \neq x$. But then $z \in E_1 \bigcup E_2 = E$ and thus we have

$$Lim(E_1) \bigcup Lim(E_2) \subset Lim(E).$$

To prove the opposite inclusion, we use the fact that $x \in Lim\,(S)$ iff there exists a sequence of elements of S, different from x, which converges to x. Let $x \in Lim(E) = Lim(E_1 \bigcup E_2))$. Then there exists a sequence $z_i \in E_1 \bigcup E_2$ such that $z_i \neq x$ and $z_i \to x$. Let us consider the two subsequences $z_{i_n} \in E_1$ and $z_{j_n} \in E_2$. At least one of these subsequences must be infinite. But every infinite subsequence of a convergent sequence $z_i \to x$ also converges to x. Thus at least one of the sequences $z_{i_n} \to x \in Lim\,(E_1)$ or $z_{j_n} \to x \in Lim\,(E_2)$.

8. Prove that if $|x| < 1$ and the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

Solution

We use the following:

(a) *Theorem on convergence radius of power series.* Any power series $\sum_{n=0}^{\infty} a_n x^n$ has a convergence radius $R \geq 0$ such that the series converges for $|x| < R$ and diverges for $|x| > R$.

(b) *Theorem on absolute convergence of power series.* Inside its radius of convergence, $|x| < R$, the power series converges absolutely.

The series $\sum_{n=0}^{\infty} a_n x^n$ with $x = 1$ becomes $\sum_{n=0}^{\infty} a_n$, which converges by assumption. Thus, $R \geq 1$ by Theorem (a). Theorem (b) then implies the result.