University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
January 12, 2009

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Andrew Knyazev, Julien Langou, Jan Mandel (Chair)

1. Let f and g be Riemann integrable on $[a, b]$ and $f \leq g$. Using the definition of Riemann integral, show that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Solution

The Riemann integral satisfies

$$\int_a^b f(x)\,dx = \inf_P U(f, P)$$

where

$$U(f, P) = \sum_{i=1}^{n} \sup_{t \in [x_{i-1}, x_i]} f(t)\,(x_i - x_{i-1})$$

and the infimum is taken over all partitions $P = \{x_0, \ldots, x_n\}$ such that

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = 1.$$

Let P be a partition. Then

$$f(t) \leq g(t), \quad \forall t \in [x_{i-1}, x_i].$$

Thus

$$U(f, P) \leq U(g, P) \text{ for any partition } P.$$

Consequently,

$$\int_a^b f(x)\,dx = \inf_P U(f, P) \leq \inf_P U(g, P) = \int_a^b g(x)\,dx.$$

2. Suppose that f_n and g_n are real functions on a set S, $f_n \to 0$ uniformly on S and $\{g_n\}$ are uniformly bounded on S. Show that $f_n g_n \to 0$ uniformly on S.

Solution

Let us denote the independent variable by $x \in S$. The definition of the uniform convergence $f_n \to 0$ states that

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n > N \quad \forall x \in S : \quad |f_n(x)| < \varepsilon.$$

The definition of the uniform boundedness of $\{g_n\}$ states that

$$\exists C > 0 \quad \forall x \in S \quad \forall n : \quad |g_n(x)| < C.$$

Thus, $\forall x \in S$ and $\forall n$ we have

$$|g_n(x) f_n(x)| \leq C \, |f_n(x)| \, .$$

We need to prove that

$$\forall \varepsilon_1 > 0 \quad \exists N_1 \quad \forall n > N_1 \quad \forall x \in S : \quad |g_n f_n(x)| < \varepsilon_1.$$

Let us set $\varepsilon = \varepsilon_1 / C$ then taking $N_1 = N$ gives $\forall n > N_1 = N \, \forall x \in S$:

$$|g_n(x) f_n(x)| < C \, |f_n(x)| < C\varepsilon = \varepsilon_1,$$

which is the required claim.

3. Determine if $d\,(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$ or not.

Solution

We will prove that d is a metric on $\mathbb{R}$ by proving that d is nonnegative, symmetric and satisfies the triangle inequality. Nonnegativity ($d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$) and symmetry ($d(x, y) = d(y, x)$) are trivially true. We focus on the triangle inequality. Take any x, y and z in $\mathbb{R}$. We want to prove that $d(x, y) \leq d(x, z) + d(z, y)$.

$$\begin{aligned} d(x, y)^2 &= |x - y| = |x - z + z - y| \leq |x - z| + |z - y| \\ &= d(x, z)^2 + d(z, y)^2 \leq d(x, z)^2 + d(z, y)^2 + 2d(x, z)d(z, y) \\ &= (d(x, z) + d(z, y))^2 \end{aligned}$$

Therefore, for any x, y and z in $\mathbb{R}$, $d(x, y) \leq d(x, z) + d(z, y)$.

4. Suppose f is a real function on $\mathbb{R}$ with bounded first derivative. For a given ε, define $g_\varepsilon(x) = x + \varepsilon f(x)$. Prove that g_ε is one to one if $\varepsilon > 0$ is small enough.

Solution

From the assumptions, there exists M such that for any x in $\mathbb{R}$, $|f'(x)| \leq M$. We will prove that for sufficiently small ε, g_ε is strictly increasing, therefore one-to-one. Indeed,

$$g'_\varepsilon(x) = (x + \varepsilon f(x))' = 1 + \varepsilon f'(x) \geq 1 - \varepsilon M,$$

so

$$g'_\varepsilon(x) > 0$$

if

$$0 < \varepsilon < 1/M.$$

5. Suppose X and Y are metric spaces, $f : X \to Y$ uniformly continous, and $\{x_n\}$ is a Cauchy sequence in X. Prove that $\{f(x_n)\}$ is Cauchy in Y.

Solution

Let $\varepsilon > 0$. Since f is uniformly continuous, $\exists\, \delta \geq 0$ such that, for all points $u \in X$ and $v \in X$ such that $d_X(u, v) \leq \delta$, we have $d_Y(f(u, v)) \leq \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence in X, $\exists\, N$ natural number such that $d_X(x_k, x_\ell) \leq \delta$ if $k \geq N$ and $\ell \geq N$. Therefore,

$$d_Y(f(x_k), f(x_\ell)) \leq \varepsilon \text{ if } k \geq N \text{ and } \ell \geq N.$$

6. Consider the set $\mathbb{Q}$ of all rational numbers as a metric space with $d(x, y) = |x - y|$, and $E = \left\{x \in \mathbb{Q} : x > 0 \text{ and } 2 < x^2 < 3\right\}$. Show that E is closed. (You can assume known that there is no $x \in \mathbb{Q}$ such that $x^2 = 2$ or $x^2 = 3$.)

Solution

Let $z \in \mathbb{Q}$ be a limit point of the set E. By definition of limit point, $\exists \{q_n\}$ such that $\forall n$, $q_n \in E$, and $q_n \to z$ in $\mathbb{Q}$. Since $\mathbb{Q} \subset \mathbb{R}$ with the same metric, it also holds $q_n \to z$ in $\mathbb{R}$. Consequently, $q_n^2 \to z^2$ in $\mathbb{R}$. Since $q_n \in E$, thus $2 < q_n^2 < 3$ for all n, it follows that $2 \le z^2 \le 3$. Since there is no $z \in \mathbb{Q}$ such that $z^2 = 2$ or $z^2 = 3$, it follows that $2 < z^2 < 3$. From $q_n \ge 0$, we have that $z \ge 0$ but $z = 0$ is not possible because $z^2 \ge 2$, so $z > 0$. Thus $z \in E$ by the definition of E.

7. Give an example of a function f on $\mathbb{R}^2$ such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point of $\mathbb{R}^2$ but f is not continuous at $(0,0)$.

Solution

Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$$

Since the denominator is nonzero for $(x,y) \neq (0,0)$, the partial derivatives exist except at $(0,0)$ by elementary rules of differentiation. At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

since $f(h,0) = 0$ for all h. Similarly, $\frac{\partial f}{\partial x}(0,0) = 0$. Thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point of $\mathbb{R}^2$. Now

$$\lim_{x \to 0} f(x,0) = \lim_{x \to 0} 0 = 0 \neq \lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2},$$

so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because if $\lim_{(x,y)\to(0,0)} f(x,y) = A$ then both limits above would have to equal to A. (Note that the statement of the problem does not say "show from the definition of limit", so we can leave it at that.)

8. Find all intervals where the series $\sum_{n=1}^{\infty} xe^{-nx}$ converges, converges absolutely, and converges uniformly.

Solution

Note that $e^{-nx} = (e^{-x})^n$, thus, for any $x > 0$, the series is a geometric series with quotient $q = e^{-x} \in (0,1)$, and so it converges absolutely,

$$\sum_{n=1}^{\infty} xe^{-nx} = \frac{xe^{-x}}{1-e^{-x}}, \; x > 0.$$

For $x = 0$, all terms of the series are zero, so the series converges absolutely also. Thus the series converges absolutely for all $x \geq 0$. For $x < 0$, $e^{-nx} \to \infty$, so the series diverges because its terms do not converge to zero. We have for $x \geq 0$,

$$0 \leq xe^{-nx} = xe^{-x}e^{-(n-1)x}$$

The function $f(x) = xe^{-x}$ is bounded on $[0,\infty)$: since $\lim_{x\to\infty} f(x) = 0$, there exists A such that $f(x) \leq 1$ for all $x > A$, and f is continuous on the compact interval $[0, A]$, thus bounded on $[0, A]$. So, let $M \geq xe^{-x}$ for all $x \in [0,\infty)$. Then for any $a > 0$,

$$\left|xe^{-nx}\right| \leq M\left(e^{-a}\right)^{n-1}, \quad \forall x \in x \in (a,\infty).$$

So by comparison with the geometric series $\sum_{n=1}^{\infty} \left(e^{-a}\right)^{n-1}$, convergence is uniform in any interval of the form $(a, +\infty)$, for any $a > 0$.

However as a gets close to 0, the quotient e^{-a} gets close to 1 so the convergence gets slower and slower. We will prove that convergence is not uniform on $[0,\infty)$. Suppose $s(x) = \sum_{n=1}^{\infty} xe^{-nx}$ uniformly on $[0,\infty)$. Since the functions $g_n(x) = xe^{-nx}$ are continuous on $[0,\infty)$, $s(x)$ is also continuous on $[0,\infty)$. But

$$\lim_{x\to0^+} s(x) = \lim_{x\to0^+} \frac{xe^{-x}}{1-e^{-x}} = \lim_{x\to0^+} \frac{xe^{-x}}{1-(1-x+o(x))} = 1 \neq s(0) = 0,$$

which is a contradiction.