

University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
January 12, 2009

Name: _____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1. _____	5. _____
2. _____	6. _____
3. _____	7. _____
4. _____	8. _____

Total _____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
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1. Let f and g be Riemann integrable on $[a, b]$ and $f \leq g$. Using the definition of Riemann integral, show that

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Solution

The Riemann integral satisfies

$$\int_a^b f(x) dx = \inf_P U(f, P)$$

where

$$U(f, P) = \sum_{i=1}^n \sup_{t \in [x_{i-1}, x_i]} f(t) (x_i - x_{i-1})$$

and the infimum is taken over all partitions $P = \{x_0, \dots, x_n\}$ such that

$$0 = x_0 < x_1 < x_2 < \dots < x_n = 1.$$

Let P be a partition. Then

$$f(t) \leq g(t), \quad \forall t \in [x_{i-1}, x_i].$$

Thus

$$U(f, P) \leq U(g, P) \text{ for any partition } P.$$

Consequently,

$$\int_a^b f(x) dx = \inf_P U(f, P) \leq \inf_P U(g, P) = \int_a^b g(x) dx.$$

2. Suppose that f_n and g_n are real functions on a set S , $f_n \rightarrow 0$ uniformly on S and $\{g_n\}$ are uniformly bounded on S . Show that $f_n g_n \rightarrow 0$ uniformly on S .

Solution

Let us denote the independent variable by $x \in S$. The definition of the uniform convergence $f_n \rightarrow 0$ states that

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n > N \quad \forall x \in S : \quad |f_n(x)| < \varepsilon.$$

The definition of the uniform boundedness of $\{g_n\}$ states that

$$\exists C > 0 \quad \forall x \in S \quad \forall n : \quad |g_n(x)| < C.$$

Thus, $\forall x \in S$ and $\forall n$ we have

$$|g_n(x)f_n(x)| \leq C |f_n(x)|.$$

We need to prove that

$$\forall \varepsilon_1 > 0 \quad \exists N_1 \quad \forall n > N_1 \quad \forall x \in S : \quad |g_n f_n(x)| < \varepsilon_1.$$

Let us set $\varepsilon = \varepsilon_1/C$ then taking $N_1 = N$ gives $\forall n > N_1 = N \quad \forall x \in S$:

$$|g_n(x)f_n(x)| < C |f_n(x)| < C\varepsilon = \varepsilon_1,$$

which is the required claim.

3. Determine if $d(x, y) = \sqrt{|x - y|}$ is a metric on \mathbb{R} or not.

Solution

We will prove that d is a metric on \mathbb{R} by proving that d is nonnegative, symmetric and satisfies the triangle inequality. Nonnegativity ($d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$) and symmetry ($d(x, y) = d(y, x)$) are trivially true. We focus on the triangle inequality. Take any x, y and z in \mathbb{R} . We want to prove that $d(x, y) \leq d(x, z) + d(z, y)$.

$$\begin{aligned}d(x, y)^2 &= |x - y| = |x - z + z - y| \leq |x - z| + |z - y| \\ &= d(x, z)^2 + d(z, y)^2 \leq d(x, z)^2 + d(z, y)^2 + 2d(x, z)d(z, y) \\ &= (d(x, z) + d(z, y))^2\end{aligned}$$

Therefore, for any x, y and z in \mathbb{R} , $d(x, y) \leq d(x, z) + d(z, y)$.

4. Suppose f is a real function on \mathbb{R} with bounded first derivative. For a given ε , define $g_\varepsilon(x) = x + \varepsilon f(x)$. Prove that g_ε is one to one if ε is small enough.

Solution

From the assumptions, there exists M such that for any x in \mathbb{R} , $|f'(x)| \leq M$. We will prove that for sufficiently small ε , g_ε is strictly increasing, therefore one-to-one. Indeed,

$$g'_\varepsilon(x) = (x + \varepsilon f(x))' = 1 + \varepsilon f'(x) \geq 1 - \varepsilon M,$$

so

$$g'_\varepsilon(x) > 0$$

if

$$0 < \varepsilon < 1/M.$$

5. Suppose X and Y are metric spaces, $f : X \rightarrow Y$ uniformly continuous, and $\{x_n\}$ is a Cauchy sequence in X . Prove that $\{f(x_n)\}$ is Cauchy in Y .

Solution

Let $\varepsilon > 0$. Since f is uniformly continuous, $\exists \delta \geq 0$ such that, for all points $u \in X$ and $v \in X$ such that $d_X(u, v) \leq \delta$, we have $d_Y(f(u), f(v)) \leq \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence in X , $\exists N$ natural number such that $d_X(x_k, x_\ell) \leq \delta$ if $k \geq N$ and $\ell \geq N$. Therefore,

$$d_Y(f(x_k), f(x_\ell)) \leq \varepsilon \text{ if } k \geq N \text{ and } \ell \geq N.$$

6. Consider the set \mathbb{Q} of all rational numbers as a metric space with $d(x, y) = |x - y|$, and $E = \{x \in \mathbb{Q} : x > 0 \text{ and } 2 < x^2 < 3\}$. Show that E is closed. (You can assume known that there is no $x \in \mathbb{Q}$ such that $x^2 = 2$ or $x^2 = 3$.)

Solution

Let $z \in \mathbb{Q}$ be a limit point of the set E . By definition of limit point, $\exists \{q_n\}$ such that $\forall n, q_n \in E$, and $q_n \rightarrow z$ in \mathbb{Q} . Since $\mathbb{Q} \subset \mathbb{R}$ with the same metric, it also holds $q_n \rightarrow z$ in \mathbb{R} . Consequently, $q_n^2 \rightarrow z^2$ in \mathbb{R} . Since $q_n \in E$, thus $2 < q_n^2 < 3$ for all n , it follows that $2 \leq z^2 \leq 3$. Since there is no $z \in \mathbb{Q}$ such that $z^2 = 2$ or $z^2 = 3$, it follows that $2 < z^2 < 3$. From $q_n \geq 0$, we have that $z \geq 0$ but $z = 0$ is not possible because $z^2 \geq 2$, so $z > 0$. Thus $z \in E$ by the definition of E .

7. Give an example of a function f on \mathbb{R}^2 such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point of \mathbb{R}^2 but f is not continuous at $(0,0)$.

Solution

Let

$$f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Since the denominator is nonzero for $(x, y) \neq (0, 0)$, the partial derivatives exist except at $(0, 0)$ by elementary rules of differentiation. At $(0, 0)$,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

since $f(h, 0) = 0$ for all h . Similarly, $\frac{\partial f}{\partial x}(0, 0) = 0$. Thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point of \mathbb{R}^2 . Now

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} 0 = 0 \neq \lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2},$$

so $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist, because if $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = A$ then both limits above would have to equal to A . (Note that the statement of the problem does not say “show from the definition of limit”, so we can leave it at that.)

8. Find all intervals where the series $\sum_{n=1}^{\infty} xe^{-nx}$ converges, converges absolutely, and converges uniformly.

Solution

Note that $e^{-nx} = (e^{-x})^n$, thus, for any $x > 0$, the series is a geometric series with quotient $q = e^{-x} \in (0, 1)$, and so it converges absolutely,

$$\sum_{n=1}^{\infty} xe^{-nx} = \frac{x}{1 - e^{-x}}, \quad x > 0.$$

For $x = 0$, all terms of the series are zero, so the series converges absolutely also. Thus the series converges absolutely for all $x \geq 0$. For $x < 0$, $e^{-nx} \rightarrow \infty$, so the series diverges because its terms do not converge to zero. We have for $x \geq 0$,

$$0 \leq xe^{-nx} = xe^{-x} e^{-(n-1)x}$$

The function $f(x) = xe^{-x}$ is bounded on $[0, \infty)$: since $\lim_{x \rightarrow \infty} f(x) = 0$, there exists A such that $f(x) \leq 1$ for all $x > A$, and f is continuous on the compact interval $[0, A]$, thus bounded on $[0, A]$. So, let $M \geq xe^{-x}$ for all $x \in [0, \infty)$. Then for any $a > 0$,

$$|xe^{-nx}| \leq M (e^{-a})^{n-1}, \quad \forall x \in x \in (a, \infty).$$

So by comparison with the geometric series $\sum_{n=1}^{\infty} (e^{-a})^{n-1}$, convergence is uniform in any interval of the form $(a, +\infty)$, for any $a > 0$.

However as a gets close to 0, the quotient e^{-a} gets close to 1 so the convergence gets slower and slower. We will prove that convergence is not uniform on $[0, \infty)$. Suppose $s(x) = \sum_{n=1}^{\infty} xe^{-nx}$ uniformly on $[0, \infty)$. Since the functions $g_n(x) = xe^{-nx}$ are continuous on $[0, \infty)$, $s(x)$ is also continuous on $[0, \infty)$. But

$$\lim_{x \rightarrow 0^+} s(x) = \lim_{x \rightarrow 0^+} \frac{x}{1 - e^{-x}} = \lim_{x \rightarrow 0^+} \frac{x}{1 - (1 - x + o(x))} = 1 \neq s(0) = 0,$$

which is a contradiction.