University of Colorado Denver
Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
January 20, 2023

Student number (not your name): ____

Exam Rules:

- This is a closed book exam. You may not use external aides during the exam, such as
  - communicating with anyone other than the exam proctor;
  - consulting the internet, textbooks, solutions of previous exams, etc.
  - using calculators or mathematical software.
- You have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Each problem is worth 20 points. The weights for each part on multi-step problems are indicated in the problem.
- Be sure to show all work that is relevant for each problem, but do not turn in scratch work.
- Justify your solutions: **cite theorems that you use**, justify that their assumptions are satisfied, provide specific counter-examples for disproof, give explanations, and show calculations for numerical computations.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask the proctor.
- This exam uses the definitions from Pugh. If you want to use definitions from Rudin, please state them and use them consistently.
- Begin each solution on a new page and write on only one side of the paper. Put your student number (not your name) and page number on the top of every page. Write legibly using a dark pencil or pen.
- In case of a major disruption due to which the exam cannot be completed, for example due to health reasons or a campus evacuation, students are entitled to a choice between acceptance of partial work and a partial new problem set, or a full new problem set.

Part 1: Solve all problems 1-4.

1. Construct a compact subset of $\mathbb{R}$ with a denumerable set of cluster points. (Definitions: y is a cluster point of A if every neighborhood of y contains an element of A besides y, or, equivalently, infinitely many points of A. Denumerable set is countable and infinite.)

2. Let (X, d) be a metric space and f and g be continuous maps $f, g : X \to \mathbb{R}$. Let E be a dense subset of X.

   (a) (10 points) Prove that $f(E)$ is dense in $f(X)$.

   (b) (10 points) If $g(x) = f(x)$ for all $x \in E$, prove that $g(x) = f(x)$ for all $x \in X$.

3. Let (a_n) and (b_n) be bounded nonnegative sequences. Prove that

$$\liminf_{n\to\infty}(a_n b_n) \geq (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n).$$

4. Let $f_n(x) = \sin(n + x)$, $x \in [0, 2\pi]$, $n = 1, 2, \ldots$. Prove that (f_n) has a pointwise convergent subsequence.

Part 2 - Solve 2 out of the following 4 problems.

5. Let $f : X \to \mathbb{R}$. Define the graph of f to be the set $G = \{(x, y) \in X \times \mathbb{R} : y = f(x)\}$.
Prove:

   (a) (10 points) If f is continuous then G is closed.

   (b) (10 points) If f is continuous and X is compact, then G is compact.

6. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Darboux integrable functions which converge pointwise to a function $f : [0,1] \to \mathbb{R}$. Prove or find a counterexample:

   The function f is Darboux integrable on $[0,1]$.

   (Darboux integral as defined in Pugh is called Riemann integral in Rudin.)

7. Define an open mapping $f : X \to Y$ to be one where $f(V)$ is open in Y whenever V is open in X. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is open and continuous then it is monotone.

8. Suppose that (X, d) is a metric space and (f_n) is a sequence of continuous functions $f_n : X \to \mathbb{R}$ convergent pointwise on X to a function f.

   (a) (10 points) Prove that if $f_n \rightrightarrows f$ on X, then for every convergent sequence $(x_n) \subset X$, $\lim_{n\to\infty} f_n(x_n) = f(x)$, where $x = \lim_{n\to\infty} x_n$.

   (b) (10 points) Is the converse true? Prove or provide a counterexample.