PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
FEBRUARY 11, 2021

Student ID: ______________________________

- The examination consists of four parts associated with four skills and content areas summarized to students via email. We summarize this below.
  - **Part 1: Skills and Content Area.** Produce straightforward proofs based on general metric space definitions for point-set topology and continuous functions that may, for establishing certain steps in the proofs, utilize standard results from either undergraduate or graduate analysis (MATH 4310 or 5070, respectively). **Students solve both problems in this part.**
  - **Part 2: Skills and Content Area.** Produce proofs involving commonly studied sequence/function spaces such as ℓ^p (for $1 \leq p \leq \infty$) or $\mathcal{C}^k(X, Y)$ for some $k \in \mathbb{N}$. **Students solve both problems in this part.**
  - **Part 3: Skills and Content Area.** Identify the correct theorem to apply to prove a result by proper verification of the theorem's hypothesis. The focus is on major theorems spanning all content including some of the major results from undergraduate analysis. Such theorems include, but are not limited to, the intermediate value theorem, the mean value theorem, the Fundamental Theorem of Calculus, the Arzelà–Ascoli theorem, and the contraction mapping theorem. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
  - **Part 4: Skills and Content Area.** Prove results requiring definitions and/or theorems for differentiation/integration. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
- Make sure to justify your solutions/proofs by citing theorems that you use, provide counter-examples with explanations, follow proper proof-writing techniques, etc.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of each piece of paper**; number all pages throughout; and, just in case, write your student ID on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end.
- Ask the proctor if you have any questions.

Examination committee: Troy Butler (chair), Burt Simon, Dmitriy Ostrovskiy

Part 1

Students should complete both problems.

(1) Let (X, d) be a metric space and $A \subset X$.

We say that $x \in X$ is a **limit point** of A if for all $\epsilon > 0$, $B_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$. Here, $A \setminus \{x\} = A \cap \{x\}^c$. The set $B_\epsilon(x)$ denotes the ball of radius ϵ centered at the point x.

(a) (10 points) Use the given definition to prove that if $A \subset X$ has a finite number of points then A has no limit points.

(b) (5 points) Give an example of (X, d) and infinite set $A \subset X$ that has no limit points. Justify your example using the given definition.

(c) (5 points) Give an example of (X, d) and infinite set $A \subset X$ such that every point of A is also a limit point of A. Justify your example using the given definition.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces.

We say that $f : A \subseteq X \to Y$ is **uniformly continuous on** A if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, z \in A$ with $d_X(x, z) < \delta$ implies $d_Y(f(x), f(z)) < \epsilon$.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $\lim_{x \to -\infty} f(x) = \alpha \in \mathbb{R}$ and $\lim_{x \to \infty} f(x) = \beta \in \mathbb{R}$. Use the given definition of uniform continuity, appropriately adapted to $\mathbb{R}$, to prove that f is uniformly continuous on $\mathbb{R}$.

Note: Appropriately adapted here simply means we use $\mathbb{R}$ with the usual metric instead of (X, d_X) and (Y, d_Y) in the definition. Also, the set A in the definition is clearly $\mathbb{R}$ in the problem statement.

Part 2

Students should complete both problems.

(3) Let $\mathcal{F}$ be a family of functions from a metric space (X, d_X) to a metric space (Y, d_Y). The family $\mathcal{F}$ is **equicontinuous** if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that for $z \in X$ with $d_X(x, z) < \delta$ implies $d_Y(f(x), f(z)) < \epsilon$ for all $f \in \mathcal{F}$.

Let (X, d_X) and (Y, d_Y) be metric spaces and $\mathcal{C}(X, Y)$ be the space of all continuous functions from X to Y. Given a sequence of functions $(f_n) \subset \mathcal{C}(X, Y)$, we say that $f_n \to f$ **uniformly** if

$$\sup_{x \in X} d_Y(f(x), f_n(x)) \to 0.$$

(a) (10 points) Prove that if $(f_n) \subset \mathcal{C}(X, Y)$ and $f_n \to f$ uniformly, then $\{f_n\}_{n \in \mathbb{N}} \cup \{f\}$ is equicontinuous using the given definition.

(b) (10 points) Show the necessity of uniform convergence by providing an example of (X, d_X), (Y, d_Y) and $(f_n) \subset \mathcal{C}(X, Y)$ that converges pointwise but not uniformly to $f : X \to Y$ such that $\{f_n\}_{n \in \mathbb{N}}$ is not equicontinuous. Make sure to justify your example.

(4) If (X, d_X) is a metric space and $S \subseteq X$, then we say that S is **dense** in X if $x \in X$ implies either $x \in S$ or x is a limit point of S. A metric space is called **separable** if it contains a countable dense subset. Examples of uncountable but separable metric spaces include $\mathbb{R}^k$ for any finite positive integer k.

Let (ℓ^∞, d) denote the usual metric space defined by bounded real-valued sequences, i.e., for all $(\xi_i) \in \ell^\infty$, then there exists $c \geq 0$ such that $\sup_{i \in \mathbb{N}} |\xi_i| \leq c$, and for each $x = (\xi_i), y = (\eta_i) \in \ell^\infty$,

$$d(x, y) := \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|.$$

Let $A := \{(\xi_i) \in \ell^\infty : \xi_i = 0 \text{ or } 1 \ \forall i \in \mathbb{N}\}$.

(a) (5 points) Prove that A is uncountable.

(b) (5 points) Prove that A has no limit points using the definition of limit points from Problem (1) of this exam.

(c) (10 points) Prove that ℓ^∞ is *not* separable.

Part 3

Students should choose one of the following two problems to complete.

(5) Let $f : [0,1] \to (0,1)$ be continuously differentiable and $\max_{0\leq x\leq 1} |f'(x)| \leq 1-\epsilon$ for some $\epsilon > 0$. Prove that f has exactly one fixed point.

(6) Let $K \in \mathcal{C}([0,1] \times [0,1])$. For $f \in \mathcal{C}([0,1])$, let

$$Tf(x) := \int_0^1 K(x,y) f(y)\, dy.$$

(a) (10 points) Prove that $Tf \in \mathcal{C}([0,1])$.

(b) (10 points) Let $\mathcal{C}([0,1])$ be equipped with the sup-norm metric. Show that $\mathcal{F} := \{Tf : \|f\|_\infty \le 1\}$ is precompact in $\mathcal{C}([0,1])$. Recall that a set is called precompact if its closure is compact.

Part 4

Students should choose one of the following two problems to complete.

(7) Suppose that for every $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that $\sup_{x \in \mathbb{R}, n \in \mathbb{N}} |f_n'(x)| = M < \infty$. Furthermore, suppose that (f_n) converges for all $x \in \mathbb{R}$ and define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$.

(a) (10 points) Show that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded on each fixed interval $[a, b]$.

(b) (10 points) Show that f is continuous on $\mathbb{R}$.

(8) The Fundamental Theorem of Calculus states that if f is a real-valued continuous function on $[a, b]$ that is differentiable on (a, b), and if f' is Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Prove the above version of the Fundamental Theorem of Calculus.