Analysis Prelim Summer 2018
PhD Program in Applied Mathematics
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Asking for the solutions of 6 out of the following 7 problems.

1. Decide if functions $f_n(x) = e^{-\left|x-\frac{1}{n}\right|n^2}$ (a) converge on $\mathbb{R}$ pointwise, (b) converge on $\mathbb{R}$ uniformly.

   Solution. For any $x \in \mathbb{R}$, we have $\lim_{n\to\infty}\left|x-\frac{1}{n}\right| = x$, thus $\lim_{n\to\infty}\left|x-\frac{1}{n}\right|n^2 = \infty$ if $x \neq 0$; for $x = 0$, we have $\lim_{n\to\infty}\left|x-\frac{1}{n}\right|n^2 = \lim_{n\to\infty}\frac{1}{n}n^2 = \infty$ also. Thus, in any case, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-\left|x-\frac{1}{n}\right|n^2} = 0$ for all x, and $f_n \to 0$ pointwise. Convergence is not uniform on $\mathbb{R}$, because $\sup_{x\in\mathbb{R}} |f_n(x) - 0| \geq f_n\left(\frac{1}{n}\right) = 1 \not\to 0$.

2. Decide if the function $f(x,y) = \frac{\sin xy}{x^2+y^2}$ can be continously extended to all of $\mathbb{R}^2$.

   Solution. The function $f(x,y)$ is continuous except at the point $(0,0)$, where it is not defined, thus the answer will be positive if and only if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then we can define $f(0,0)$ as its value. If this limit exists, then it equals to the limit along any line, $\lim_{\substack{(x,y)\to(0,0)\\(x,y)=t(a,b)}} f(x,y)$ for any $(a,b) \neq (0,0)$. But

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)=t(0,1)}} f(x,y) = \lim_{y\to 0} f(0,y) = 0$$

   while

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)=t(1,1)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{\sin t^2}{t^2+t^2} = \frac{1}{2} \neq 0.$$

   Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, and continuous extension of f on all of $\mathbb{R}^2$ is not possible.

3. Let (X,d) be a metric space.

   (a) Prove that d is a continuous real-valued function on the product metric space $(X \times X, d_{X\times X})$ where $d_{X\times X}$ is a natural product metric induced by d.

   (b) Give an example of (X,d) and complete non-empty subsets $A, B \subset X$ such that there do not exist $a_0 \in A$ and $b_0 \in B$ such that

$$d(a_0, b_0) = \inf\{d(a,b) \,:\, a \in A, b \in B\}$$

   Solution.

   (a) Estimate

$$\begin{aligned} d(x_1,y_1) - d(x_2,y_2) &= d(x_1,y_1) - d(x_2,y_1) + d(x_2,y_1) - d(x_2,y_2) \\ &\leq d(x_1,x_2) + d(y_1,y_2) \end{aligned} \tag{1}$$

because

$$
\begin{aligned}
d(x_2, y_1) &\le d(x_2, x_1) + d(x_1, y_1) \\
d(x_2, y_1) - d(x_1, y_1) &\le d(x_2, x_1)
\end{aligned}
$$

and, exchanging the roles of x and y and using symmetry of metric,

$$d(x_2, y_1) - d(x_2, y_2) \le d(y_1, y_2).$$

Swapping (x_1, y_1) and (x_2, y_2) in (1), we have also

$$d(x_2, y_2) - d(x_1, y_1) \le d(x_1, x_2) + d(y_1, y_2),$$

thus

$$
\begin{aligned}
|d(x_1, y_1) - d(x_2, y_2)| &\le d(x_1, x_2) + d(y_1, y_2) \\
&= d_{X \times X}((x_1, x_2), (y_1, y_2)).
\end{aligned}
$$

(b) Consider $X = \mathbb{R}^2$ equipped with the euclidean metric

$$d((s,t), (u,v)) = \sqrt{|s-u|^2 + |t-v|^2}.$$

Define $A = \{(s, 0) \in C\}$ and $B = \left\{\left(s, \frac{1}{s}\right) \in C\right\}$, where $C = \{(s,t) : s \ge 1\}$. The set C is closed, thus complete subset of $\mathbb{R}^2$, and its subsets A and B are also closed and thus complete, because they are inverse images under continuous mappings of closed sets,

$$A = f^{-1}(\{0\}), \quad f : (s,t) \mapsto t$$

and

$$B = g^{-1}(\{1\}) \quad g : (s,t) = st.$$

We have

$$\inf\{d(a,b) \,:\, a \in A, b \in B\} \le d\left((s, 0), \left(s, \frac{1}{s}\right)\right) = \frac{1}{s}$$

for all $s \ge 1$, thus

$$\inf\{d(a,b) \,:\, a \in A, b \in B\} = 0.$$

But $A \cap B = \emptyset$, so there do not exist $a_0 \in A$ and $b_0 \in B$ such that

$$d(a_0, b_0) = 0$$

which would require $a_0 = b_0$.

4. We say that two metrics d_1 and d_2 defined on the same space X are equivalent if there exists real numbers $c_1 > 0$ and $c_2 > 0$ such that for every $x, y \in X$,

$$c_1 d_1(x,y) \le d_2(x,y) \le c_2 d_1(x,y).$$

(a) Prove that if d_1 and d_2 are equivalent metrics, then a sequence $(x_n) \subset X$ converges to x in (X, d_2) if and only if $(x_n) \subset X$ converges to x in (X, d_1).

(b) Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$. For any $f, g \in C([0,1])$, let d_I denote the *integral* metric defined by

$$d_I(f,g) = \int_0^1 |f(x) - g(x)|\, dx,$$

and d_S denote the *supremum* metric defined by

$$d_S(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|\,.$$

Prove that the metrics d_I and d_S are *not* equivalent.

Solution.

(a) Suppose $x_n \to x$ in (X, d_1), or equivalently, $d_1(x_n, x) \to 0$. Then

$$0 \le d_2(x_n, x) \le c_2 d_1(x_n, x) \to 0$$

so $d_2(x_n, x) \to 0$ by the squeeze theorem. Suppose that $x_n \to x$ in (X, d_2), then

$$0 \le d_1(x_n, x) \le \frac{1}{c_1} d_2(x_n, x) \to 0$$

and so $d_1(x_n, x) \to 0$ by the squeeze theorem.

(b) Choose $f(x) = 0$ for all x and, for $n \ge 2$, $f_n(x)$ piecewise linear given by the values

$$f_n(0) = 0, \quad f_n\left(\frac{1}{n}\right) = 1, \quad f_n\left(\frac{2}{n}\right) = 0, \quad f_n(1) = 0.$$

Then,

$$d_I(f_n, f) = \int_0^1 |f_n(x)|\, dx = \frac{1}{n} \to \infty$$

while

$$d_S(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1$$

By part 4a, the metrics d_I and d_S are not equivalent because $f_n \to f$ in d_I but not in d_S.

5. Let $\mathcal{F}$ be a bounded subset of $C([a,b])$ with the supremum metric and

$$A = \left\{ F(x) = \int_a^x f(t)\, dt \;:\; f \in \mathcal{F} \right\}.$$

Prove that the closure $\overline{A}$ of A is a compact subset of $C([a,b])$.

Solution. Since $\mathcal{F}$ is bounded, there is M such that for all $f \in \mathcal{F}$ and all $x \in [a,b]$, it holds that $|f(x)| \le M$. Let a sequence $\{U_n\} \subset \overline{A}$. Then, for every n, there exists $F_n \in A$ such that $d(U_n, F_n) < \frac{1}{n}$, and $F_n(x) = \int_a^x f_n(t)\, dt \;:\; f_n \in \mathcal{F}$. For any $x \in [a,b]$, it holds that

$$|F_n(x)| = \left| \int_a^x f_n(t)\, dt \right| \le (b-a)\, M,$$

thus $\{F_n\}$ is uniformly bounded. Similarly, for any $x, y \in [a, b]$, it holds that

$$|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t)\, dt \right| \leq |x - y| M,$$

thus the set $\{F_n\}$ is equicontinuous. Since $[a, b]$ is compact, by the Arzèla-Ascoli theorem, there exists a subsequence $\{F_{n_k}\}$ that is uniformly convergent, that is, for some $F \in C([a, b])$, $d(F_{n_k}, F) \to 0$ as $k \to \infty$. By the triangle inequality,

$$d(U_{n_k}, F) \leq d(U_{n_k}, F_{n_k}) + d(F_{n_k}, F) \to 0.$$

Since the closure of a set is closed, $F \in \overline{A}$. Thus, any sequence in $\overline{A}$ has a subsequence convergent in $\overline{A}$, so $\overline{A}$ is compact.

6. Let (X, d) be a complete metric space, and $A \subset X$, equiped with the distance function d restricted to $A \times A$, denoted by d_A. Prove that the space (A, d_A) is complete if and only if A is closed in (X, d).

   Solution. Suppose (A, d_A) is complete. Let $\{x_n\} \subset A$ converge to x in X.Then $\{x_n\}$ is Cauchy in A, consequently Cauchy in X. Since X is complete, there exists a limit y, $x_n \to y$ in X. Since limit is unique, $y = x$. Thus A is closed subset of X.

   Suppose A is closed in X. Let $\{x_n\} \subset A$ be Cauchy (A, d_A). Then $\{x_n\}$ is Cauchy (X, d), and since X is complete, there is a limit x, $x_n \to x$ in X. Since A is closed, $x \in A$. Since d and d_A coincide on A, $x_n \to x$ in X and $x \in A$ imply $x_n \to x$ in A.

7. Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

   (a) (5 points) Decide for which real numbers x the series converges.

   (b) (15 points) Decide on which intervals the series converges uniformly.

   Solution.

   (a) Since $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1$, the radius of convergence is $R = \frac{1}{1}$. That is, the series converges absolutely for all $-1 < x < 1$ and diverges for all $x < -1$ and $x > 1$. For $x = 1$, the series diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which is known to be divergent. For $x = -1$, the series converges, because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ with $a_n = \frac{1}{n} \searrow 0$, which converges by the alternating series theorem.

   (b) (5 points) Since power series at 0 converges uniformly on every interval $[-a, a]$, $a < R$, the series converges uniformly on all intervals $[-a, a]$, $a < 1$.

   It remains to consider the uniform convergence near the endpoints.

   (5 points) Convergence of the series is not uniform on any interval $(a, 1)$, $a > 0$: If convergence were uniform, there would exist N such that the partial sum $s_N(x) = \sum_{n=1}^{N} \frac{x^n}{n}$

satisfies $|f(x) - s_N(x)| < 1$ for all $x \in (a, 1)$. But $s_N(x)$ is a polynomial, thus a bounded function, while

$$\lim_{x \to 1_-} f(x) \geq \lim_{x \to 1_-} s_m(x) = \sum_{n=1}^{m} \frac{1}{n} \to \infty \text{ as } m \to \infty,$$

thus f is not bounded on $(a, 1)$, contradiction.

(5 points) Convergence of the series is uniform on the interval $[-1, 0]$: Since, for a fixed $x \in [-1, 0]$, the sum $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is alternating series with monotonically decreasing absolute values of its terms, we have

$$s_{2k-1}(x) \leq f(x) \leq s_{2k}(x), \; s_{2k}(x) - s_{2k-1}(x) \leq \frac{1}{2k}$$

thus

$$|s_m(x) - f(x)| \leq \frac{1}{m}$$

for all m and all $x \in [-1, 0]$, which proves uniform convergence on $[-1, 0]$.

In conclusion, convergence of the power series is uniform on all intervals $[-1, a)$, $a < 1$, but not on any interval with end point 1.

Coverage and syllabus check by problem number:

1. uniform convergence Rudin ch. 7
2. multivariate continuity ch. 9; version of problem 9.6, standard undergraduate real 2 (or calculus)
3. definition of infimum (ch. 1), definition compactness (ch. 2)
4. straightforward by definition of convergence in metric space (ch. 2)
5. Arzela-Ascoli theorem, sequentially compact (ch. 7, exercise 2.26)
6. complete metric space (ch. 2)
7. power series, radius of convergence, uniform convergence, alternating series. This is a version of the standard capstone problem of power series, which involves computing the sum of the series by differentiation or integration and Abel's theorem.

Note: lim sup and lim inf not covered this time.