Asking for the solutions of 6 out of the following 7 problems.

1. Decide if functions \( f_n(x) = e^{-|x-\frac{1}{n}|n^2} \) converge on \( \mathbb{R} \) pointwise, (b) converge on \( \mathbb{R} \) uniformly.

**Solution.** For any \( x \in \mathbb{R} \), we have \( \lim_{n \to \infty} |x - \frac{1}{n}| = x \), thus \( \lim_{n \to \infty} |x - \frac{1}{n}| n^2 = \infty \) if \( x \neq 0 \); for \( x = 0 \), we have \( \lim_{n \to \infty} |x - \frac{1}{n}| n^2 = \lim_{n \to \infty} \frac{1}{n} n^2 = \infty \) also. Thus, in any case, \( \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} e^{-|x-\frac{1}{n}|n^2} = 0 \) for all \( x \), and \( f_n \to 0 \) pointwise. Convergence is not uniform on \( \mathbb{R} \), because \( \sup_{x \in \mathbb{R}} |f_n(x) - 0| \geq f_n \left( \frac{1}{n} \right) = 1 \not\to 0 \).

2. Decide if the function \( f(x, y) = \frac{\sin xy}{x^2+y^2} \) can be continuously extended to all of \( \mathbb{R}^2 \).

**Solution.** The function \( f(x, y) \) is continuous except at the point \((0,0)\), where it is not defined, thus the answer will be positive if and only if \( \lim_{(x,y) \to (0,0)} f(x, y) \) exists, then we can define \( f(0,0) \) as its value. If this limit exists, then it equals to the limit along any line, \( \lim_{(x,y) \to (0,0)} f(x, y) \) for any \((a,b) \neq (0,0)\). But
\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(x,y) \to (0,1)} f(x, y) = 0
\]
while
\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{t \to 0} f(t, t) = \lim_{t \to 0} \frac{\sin t^2}{t^2} = \frac{1}{2} \neq 0.
\]
Thus, \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist, and continuous extension of \( f \) on all of \( \mathbb{R}^2 \) is not possible.

3. Let \((X, d)\) be a metric space.

(a) Prove that \( d \) is a continuous real-valued function on the product metric space \((X \times X, d_{X \times X})\) where \( d_{X \times X} \) is a natural product metric induced by \( d \).

(b) Give an example of \((X, d)\) and complete non-empty subsets \( A, B \subset X \) such that there do not exist \( a_0 \in A \) and \( b_0 \in B \) such that
\[
d(a_0, b_0) = \inf \{d(a, b) : a \in A, b \in B\}
\]

**Solution.**

(a) Estimate
\[
d(x_1, y_1) - d(x_2, y_2) = d(x_1, y_1) - d(x_2, y_1) + d(x_2, y_1) - d(x_2, y_2) \\
\leq d(x_1, x_2) + d(y_1, y_2)
\]
because
\[ d(x_2, y_1) \leq d(x_2, x_1) + d(x_1, y_1) \]
\[ d(x_2, y_1) - d(x_1, y_1) \leq d(x_2, x_1) \]
and, exchanging the roles of \( x \) and \( y \) and using symmetry of metric,
\[ d(x_2, y_1) - d(x_2, y_2) \leq d(y_1, y_2). \]
Swapping \((x_1, y_1)\) and \((x_2, y_2)\) in (1), we have also
\[ d(x_2, y_2) - d(x_1, y_1) \leq d(x_1, y_1) + d(y_1, y_2), \]
thus
\[ |d(x_1, y_1) - d(x_2, y_2)| \leq d(x_1, x_2) + d(y_1, y_2), \]
\[ = d_{X \times X}((x_1, x_2), (y_1, y_2)). \]
(b) Consider \( X = \mathbb{R}^2 \) equipped with the euclidean metric
\[ d((s, t), (u, v)) = \sqrt{|s - u|^2 + |t - v|^2}. \]
Define \( A = \{(s, 0) \in C\} \) and \( B = \{(s, \frac{1}{s}) \in C\}, \) where \( C = \{(s, t) : s \geq 1\}. \) The set \( C \)
is closed, thus complete subset of \( \mathbb{R}^2, \) and its subsets \( A \) and \( B \) are also closed and thus
complete, because they are inverse images under continuous mappings of closed sets,
\[ A = f^{-1}(\{0\}) , \; f : (s, t) \mapsto t \]
and
\[ B = g^{-1}(\{1\}) \; g : (s, t) = st. \]
We have
\[ \inf \{d(a, b) : a \in A, b \in B\} \leq d\left((s, 0), \left(s, \frac{1}{s}\right)\right) = \frac{1}{s} \]
for all \( s \geq 1, \) thus
\[ \inf \{d(a, b) : a \in A, b \in B\} = 0. \]
But \( A \cap B = \emptyset, \) so there do not exist \( a_0 \in A \) and \( b_0 \in B \) such that
\[ d(a_0, b_0) = 0 \]
which would require \( a_0 = b_0. \)
4. We say that two metrics \( d_1 \) and \( d_2 \) defined on the same space \( X \) are equivalent if there exists
real numbers \( c_1 > 0 \) and \( c_2 > 0 \) such that for every \( x, y \in X, \)
\[ c_1d_1(x, y) \leq d_2(x, y) \leq c_2d_1(x, y). \]
(a) Prove that if \( d_1 \) and \( d_2 \) are equivalent metrics, then a sequence \((x_n) \subset X\) converges to \( x \)
in \((X, d_2)\) if and only if \((x_n) \subset X\) converges to \( x \) in \((X, d_1)\).
(b) Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$. For any $f, g \in C([0,1])$, let $d_I$ denote the integral metric defined by

$$d_I(f, g) = \int_0^1 |f(x) - g(x)| \, dx,$$

and $d_S$ denote the supremum metric defined by

$$d_S(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Prove that the metrics $d_I$ and $d_S$ are not equivalent.

Solution.

(a) Suppose $x_n \to x$ in $(X, d_1)$, or equivalently, $d_1(x_n, x) \to 0$. Then

$$0 \leq d_2(x_n, x) \leq c_2 d_1(x_n, x) \to 0$$

so $d_2(x_n, x) \to 0$ by the squeeze theorem. Suppose that $x_n \to x$ in $(X, d_2)$, then

$$0 \leq d_1(x_n, x) \leq \frac{1}{c_1} d_2(x_n, x) \to 0$$

and so $d_1(x_n, x) \to 0$ by the squeeze theorem.

(b) Choose $f(x) = 0$ for all $x$ and, for $n \geq 2$, $f_n(x)$ piecewise linear given by the values

$$f_n(0) = 0, \quad f_n\left(\frac{1}{n}\right) = 1, \quad f_n\left(\frac{2}{n}\right) = 0, \quad f_n(1) = 0.$$

Then,

$$d_I(f_n, f) = \int_0^1 |f_n(x)| \, dx = \frac{1}{n} \to \infty$$

while

$$d_S(f_n, f) = \sup_{x \in [0,1]} |f_n(x)| = 1$$

By part 4a, the metrics $d_I$ and $d_S$ are not equivalent because $f_n \to f$ in $d_I$ but not in $d_S$.

5. Let $\mathcal{F}$ be a bounded subset of $C([a, b])$ with the supremum metric and

$$A = \left\{ F(x) = \int_a^x f(t) \, dt : f \in \mathcal{F} \right\}.$$

Prove that the closure $\overline{A}$ of $A$ is a compact subset of $C([a, b])$.

Solution. Since $\mathcal{F}$ is bounded, there is $M$ such that for all $f \in \mathcal{F}$ and all $x \in [a, b]$, it holds that $|f(x)| \leq M$. Let a sequence $\{U_n\} \subset A$. Then, for every $n$, there exists $F_n \in A$ such that $d(U_n, F_n) < \frac{1}{n}$, and $F_n(x) = \int_a^x f_n(t) \, dt : f_n \in \mathcal{F}$. For any $x \in [a, b]$, it holds that

$$|F_n(x)| = \left| \int_a^x f_n(t) \, dt \right| \leq (b-a) M,$$
thus \( \{F_n\} \) is uniformly bounded. Similarly, for any \( x, y \in [a, b] \), it holds that
\[
|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t) \, dt \right| \leq |x - y| \cdot M,
\]
thus the set \( \{F_n\} \) is equicontinuous. Since \([a, b]\) is compact, by the Arzela-Ascoli theorem, there exists a subsequence \( \{F_{n_k}\} \) that is uniformly convergent, that is, for some \( F \in C([a, b]) \),
\[
d(F_{n_k}, F) \to 0 \quad \text{as} \quad k \to \infty.
\]
By the triangle inequality,
\[
d(U_{n_k}, F) \leq d(U_{n_k}, F_{n_k}) + d(F_{n_k}, F) \to 0.
\]
Since the closure of a set is closed, \( F \in \overline{A} \). Thus, any sequence in \( \overline{A} \) has a subsequence convergent in \( \overline{A} \), so \( \overline{A} \) is compact.

6. Let \((X, d)\) be a complete metric space, and \( A \subset X \), equiped with the distance function \( d \) restricted to \( A \times A \), denoted by \( d_A \). Prove that the space \((A, d_A)\) is complete if and only if \( A \) is closed in \((X, d)\).

**Solution.** Suppose \((A, d_A)\) is complete. Let \( \{x_n\} \subset A \) converge to \( x \) in \( X \). Then \( \{x_n\} \) is Cauchy in \( A \), consequently Cauchy in \( X \). Since \( X \) is complete, there exists a limit \( y, x_n \to y \) in \( X \). Since limit is unique, \( y = x \). Thus \( A \) is closed subset of \( X \).

Suppose \( A \) is closed in \( X \). Let \( \{x_n\} \subset A \) be Cauchy \((A, d_A)\). Then \( \{x_n\} \) is Cauchy \((X, d)\), and since \( X \) is complete, there is a limit \( x, x_n \to x \) in \( X \). Since \( A \) is closed, \( x \in A \). Since \( d \) and \( d_A \) coincide on \( A \), \( x_n \to x \) in \( X \) and \( x \in A \) imply \( x_n \to x \) in \( A \).

7. Consider the power series \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \)

(a) (5 points) Decide for which real numbers \( x \) the series converges.
(b) (15 points) Decide on which intervals the series converges uniformly.

**Solution.**

(a) Since \( \lim_{n \to \infty} \left( \frac{1}{n} \right)^{1/n} = 1 \), the radius of convergence is \( R = \frac{1}{e} \). That is, the series converges absolutely for all \(-1 < x < 1\) and diverges for all \( x < -1 \) and \( x > 1 \). For \( x = 1 \), the series diverges, because \( \sum_{n=1}^{\infty} \frac{1}{n} \) is the harmonic series, which is known to be divergent. For \( x = -1 \), the series converges, because \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) is of the form \( \sum_{n=1}^{\infty} (-1)^n a_n \) with \( a_n = \frac{1}{n} \searrow 0 \), which converges by the alternating series theorem.

(b) (5 points) Since power series at 0 converges uniformly on every interval \([-a, a], a < R\), the series converges uniformly on all intervals \([-a, a], a < 1\).

It remains to consider the uniform convergence near the endpoints.

(5 points) Convergence of the series is not uniform on any interval \((a, 1), a > 0\): If convergence were uniform, there would exist \( N \) such that the partial sum \( s_N(x) = \sum_{n=1}^{N} \frac{x^n}{n} \)

satisfies $|f(x) - s_N(x)| < 1$ for all $x \in (a, 1)$. But $s_N(x)$ is a polynomial, thus a bounded function, while

$$\lim_{x \to 1^-} f(x) \geq \lim_{x \to 1^-} s_m(x) = \sum_{n=1}^{m} \frac{1}{n} \to \infty \text{ as } m \to \infty,$$

thus $f$ is not bounded on $(a, 1)$, contradiction.

(5 points) Convergence of the series is uniform on the interval $[-1, 0]$: Since, for a fixed $x \in [-1, 0]$, the sum $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is alternating series with monotonically decreasing absolute values of its terms, we have

$$s_{2k-1}(x) \leq f(x) \leq s_{2k}(x), \ s_{2k}(x) - s_{2k-1}(x) \leq \frac{1}{2k},$$

thus

$$|s_m(x) - f(x)| \leq \frac{1}{m}$$

for all $m$ and all $x \in [-1, 0]$, which proves uniform convergence on $[-1, 0]$.

In conclusion, convergence of the power series is uniform on all intervals $[-1, a), a < 1$, but not on any interval with end point 1.

Coverage and syllabus check by problem number:
1. uniform convergence Rudin ch. 7
2. multivariate continuity ch. 9; version of problem 9.6, standard undergraduate real 2 (or calculus)
3. definition of infimum (ch. 1), definition compactness (ch. 2)
4. straightforward by definition of convergence in metric space (ch. 2)
5. Arzela-Ascoli theorem, sequentially compact (ch. 7, exercise 2.26)
6. complete metric space (ch. 2)
7. power series, radius of convergence, uniform convergence, alternating series. This is a version of the standard capstone problem of power series, which involves computing the sum of the series by differentiation or integration and Abel’s theorem.

Note: lim sup and lim inf not covered this time.