PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JANUARY 25, 2019

Name: ________________________________

• The examination consists of 6 problems.
• Each problem is worth 20 points. Unless specified otherwise, numbered parts of a
  problem have equal weight.
• Justify your solutions: cite theorems that you use, provide counter-examples, give
  explanations.
• Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
• Please begin solution to every problem on a new page; write only on one side of
  paper; number all pages throughout; and, just in case, write your name on every
  page.
• Do not submit scratch paper or multiple alternative solutions. If you do, we will
  grade the first solution to its end and we will not attempt to fish for the truth.
• Ask the proctor if you have any questions.

  Good luck!

1. _______
2. _______
3. _______
4. _______
5. _______
6. _______

Total ______

Examination committee: Jan Mandel, Dmitriy Ostrovskiy, Burt Simon (chair).
(1) Let \( \{x_n\} \) be a sequence of real numbers. Prove that \( \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \). 

Hint: You can use the fact that the infimum of a set is less than or equal to the supremum.
(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable, and suppose \( f'(x) > 0, \ x \in (a, b) \). Prove that \( f \) is strictly increasing on \([a, b]\).
(3) Let \( \{ f_n \} \) be a sequence of real-valued functions on \( D \subset \mathbb{R} \) such that \( |f_n(x)| \leq M_n < \infty \) for all \( n \) and all \( x \in D \).

(a) Prove that if \( \sum_{n=1}^{\infty} M_n \) converges, then \( \sum_{n=1}^{\infty} f_n \) converges uniformly on \( D \). (This is the Weierstrass M-test.)

(b) Show that the converse is not true by constructing a counterexample.
(4) Let \( \{x_n\} \) be a bounded sequence of real numbers.
(a) Prove that \( x_n \rightarrow x \) if and only if every convergent subsequence of \( \{x_n\} \) converges to \( x \).
(b) Find a counterexample to part (a) if the sequence is not bounded.
(5) Let \((X, d)\) be a metric space, and let \(A \subset X\). Define \(\partial A\) (the boundary of \(A\)) to be the set of all points in \(X\) for which every neighborhood contains at least one point in \(A\) and at least one point in \(A^c\). Prove that \(\partial A = \bar{A} \cap \bar{A}^c\).
(6) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on every interval $[0, t], \ t < \infty$ and define

$$I = \lim_{t \to \infty} \int_{0}^{t} f(x) \, dx$$

if the limit exists. We say that $f$ is absolutely integrable on $[0, \infty)$ if

$$\lim_{t \to \infty} \int_{0}^{t} |f(x)| \, dx < \infty.$$