University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 9, 2017

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Consider the map $\varphi : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$ where

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a+b-c & c-d \\ 2a+c & a-b+d \end{pmatrix}$$

a) Is φ bijective? Prove your claim.

b) Compute $(a,b,c,d)^T$ such that $\varphi((a,b,c,d)^T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or decide that this is not possible.

Solution: Recall the isomorphism of all 4-dimensional vector spaces over $\mathbb{R}$. Instead of φ, we consider the map $\bar{\varphi} : \mathbb{R}^4 \to \mathbb{R}^4$ with $(a,b,c,d)^T \mapsto (a+b-c, c-d, 2a+c, a-b+d)^T$. All claims for $\bar{\varphi}$ can be readily transferred to φ. But now we have $\bar{\varphi}((a,b,c,d)^T) = A(a,b,c,d)^T$ with $A := \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$.

a) As $\bar{\varphi}$ is linear and maps from $\mathbb{R}^4$ to $\mathbb{R}^4$, $\bar{\varphi}$ is bijective if it is injective, i.e. if $\ker(A) = \{0\}$. Thus it remains to check whether A has full rank:

$$\begin{array}{c} \\ III - 2I \\ II \\ IV - 2I \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ -1/2II \\ \\ IV - II \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ \\ IV + 2III \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Thus $\bar{\varphi}$ and then also φ are bijective.

b) We use $(1,0,0,1)^T$ as right-hand side for the above computation and obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & -3/2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

This gives $d = -2$, $c = d = -2$, $b = 1 + 3/2c = -2$ and $a = 1 - b + c = 1$.

Problem 2

Let $x_1, \ldots, x_n \in \mathbb{C}$ with $n \geq 2$. Then

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

is called a *Vandermonde Matrix*. Prove that

$$\det(V(x_1, \ldots, x_n)) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Solution: We prove the claim by induction over n. For $n = 2$, we see $\det(V(x_1, x_2)) = 1 \cdot x_2 - x_1 \cdot 1 = x_2 - x_1 = \prod_{1 \leq i < j \leq 2}(x_j - x_i)$. Now assume that the given formula holds for all $n \in [n_0 - 1]$, $n_0 \geq 2$. Then we will prove that it also holds for $n = n_0$. To do so, we subtract the x_1-multiple of each column from the column following it and obtain

$$\begin{aligned} \det(V(x_1, \ldots, x_{n_0})) &= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n_0-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n_0} - x_1 & (x_{n_0} - x_1)x_{n_0} & \cdots & (x_{n_0} - x_1)x_{n_0}^{n_0-2} \end{pmatrix} \\ &= \det \begin{pmatrix} x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n_0-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_0} - x_1 & (x_{n_0} - x_1)x_{n_0} & \cdots & (x_{n_0} - x_1)x_{n_0}^{n_0-2} \end{pmatrix} \\ &= (x_2 - x_1)(x_3 - x_1) \cdots (x_{n_0} - x_1) \cdot \det \begin{pmatrix} 1 & x_2 & \cdots & x_2^{n_0-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n_0} & \cdots & x_{n_0}^{n_0-2} \end{pmatrix} \\ &= \prod_{j=2}^{n_0} (x_j - x_1) \cdot \det(V(x_2, \ldots, x_{n_0})) \end{aligned}$$

By our induction assumption, we know that $\det(V(x_2, \ldots, x_{n_0})) = \prod_{2 \leq i < j \leq n_0}(x_j - x_i)$, so we obtain $\det(V(x_1, \ldots, x_{n_0})) = \prod_{j=2}^{n_0}(x_j - x_1) \cdot \det(V(x_2, \ldots, x_{n_0})) = \prod_{1 \leq i < j \leq n_0}(x_j - x_i)$.

Problem 3

Let $V = \mathbb{R}[x]_{\leq 2} = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the space of polynomials of degree ≤ 2. Let $u_1 = 1$, $u_2 = x$, $u_3 = x^2$ and $b_1 = 1$, $b_2 = x + 1$, $b_3 = x^2 + x + 1$. Then $E = \{u_1, u_2, u_3\}$ and $B = \{b_1, b_2, b_3\}$ are bases of V. Further, let $C = \{c_1, c_2, c_3\}$ be a basis that is not explicitly known. The only available information is the basis transformation matrix

$$S_{E,C} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix}.$$

Notation. $S_{E,C}$ is the representation matrix of the identity $id\colon \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto x$, where the coordinates of x with respect to E are given before the mapping, and the coordinates with respect to C are given after the mapping.

a) Compute the basis transformation matrices $S_{E,B}$ and $S_{B,E}$.

b) Let $y := (1, 2, -1)^T \in \mathbb{R}^3$ be the coordinate vector of a vector $v \in V$ with respect to basis B. What is the coordinate vector of v with respect to E? Further, compute v explicitly.

c) What is the coordinate vector of $w := 2 - 3x + x^2 \in V$ with respect to C? Further, write w as a linear combination of the elements of C.

d) Compute the basis C.

Solution:

a) $b_1 = u_1$, $b_2 = u_1 + u_2$, $b_3 = u_1 + u_2 + u_3$. Thus

$$S_{B,E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Due to $S_{E,B} = S_{B,E}^{-1}$, we could obtain $S_{E,B}$ by inverting $S_{B,E}$. Alternatively, we can represent the basis vectors as linear combinations: $u_1 = b_1$, $u_2 = -b_1 + b_2$, $u_3 = -b_2 + b_3$ and thus

$$S_{E,B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) $y = (1, 2, -1)^T$ are the coordinates of v with respect to B. Thus we obtain coordinates for v with respect to E through $S_{B,E} y = (2, 1, -1)^T$.

Explicitly: $v = b_1 + 2b_2 - b_3 = 1 + 2(x + 1) - (x^2 + x + 1) = 2 + x - x^2 = 2u_1 + u_2 - u_3$.

c) $w = 2 - 3x + x^2 = 2u_1 - 3u_2 + u_3$, so w has coordinates $(2, -3, 1)^T$ with respect to E. The coordinates with respect to C are obtained by $S_{E,C}(2, -3, 1)^T = (1, -3, 2)^T$, which gives $w = c_1 - 3c_2 + 2c_3$.

d) We first compute $S_{C,E}$:

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III - I \\
I - II - 2III \\
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\end{array}
\begin{array}{l}
\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}
\end{array}$$

Thus $S_{C,E} = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. The columns of $S_{C,E}$ give C: $c_1 = 3u_1 - u_3 = 3 - x^2$, $c_2 = -u_1 + u_2 = x - 1$, $c_3 = -2u_1 + u_3 = x^2 - 2$.

Problem 4

Two $n \times n$ real matrices A and B are called *simultaneously diagonalizable* if there is an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $S^{-1}AS$ and $S^{-1}BS$ both are diagonal matrices. Let A and B be two $n \times n$ real matrices. Prove:

a) If A and B are simultaneously diagonalizable, then $AB = BA$.

b) If $AB = BA$ and if A has n different eigenvalues, then A and B are simultaneously diagonalizable.

Solution:

a) Let $D_A = S^{-1}AS$ and $D_B = S^{-1}BS$ be the two diagonal matrices. Then $D_A D_B = D_B D_A$ and thus

$$AB = SD_AS^{-1}SD_BS^{-1} = SD_AD_BS^{-1} = SD_BD_AS^{-1} = SD_BS^{-1}SD_AS^{-1} = BA.$$

b) Since A has n different eigenvalues, A is diagonalizable, i.e. there is an invertible $n\times n$ real matrix S, such that $D_A = S^{-1}AS = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix of eigenvalues $\lambda_1, \dots, \lambda_n$ of A.

We define $D_B := S^{-1}BS$. Since $AB = BA$, we obtain $D_AD_B = S^{-1}ASS^{-1}BS = S^{-1}ABS = S^{-1}BAS = S^{-1}BSS^{-1}AS = D_BD_A$. Let $D_AD_B = (c_{ij})$ and $D_BD_A = (d_{ij})$. Then $c_{ii} = d_{ii}$ is the λ_i-multiple of the i, j entry of D_B, but for $i \neq j$, c_{ij} is the λ_i-multiple of the i, j entry of D_B, while d_{ij} is the λ_j-multiple of the i, j entry of D_B. As all the eigenvalues of A are different and as $D_AD_B = D_BD_A$, D_B must be a diagonal matrix.

Problem 5

Let $A \in \mathbb{R}^{3\times 3}$ be an unknown matrix, and let

$$v_1 := \begin{pmatrix}1\\0\\2\end{pmatrix}, \quad v_2 := \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad v_3 := \begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathbb{R}^3.$$

Further let $S := (v_1|v_2|v_3) \in \mathbb{R}^{3\times 3}$ be the real 3×3 matrix with columns v_1, v_2, v_3. Finally, let

$$\ker(A + 2I_3) = \operatorname{span}\langle v_1\rangle \quad \text{and} \quad \ker(A - I_3) = \operatorname{span}\langle v_2, v_3\rangle,$$

where I_3 is the 3×3 identity matrix.

a) Prove that A is diagonalizable and give the characteristic polynomial χ_A in factorized form. Further, give all eigenvalues of A with their geometric and algebraic multiplicities. Finally, give the minimal polynomial m_A of A.

b) Compute the matrix A.

Solution:

a) The geometric multiplicities add up to 3, so A is diagonalizable. We obtain $\chi_A(x) = (x+2)(x-1)^2$ and the eigenvalues are -2 (with geometric and algebraic multiplicity 1) and 1 (with geometric and algebraic multiplicity 2). As A is diagonalizable, the minimal polynomial m_A splits (i.e. it is the product of linear factors). It takes the form $m_A(x) = (x+2)(x-1)$.

b) As A is diagonalizable, S is invertible. First, we compute S^{-1}.

$$\begin{bmatrix}1&0&1&1&0&0\\0&1&1&0&1&0\\2&0&1&0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0&-1&0&1\\0&1&1&0&1&0\\0&0&1&2&0&-1\end{bmatrix} \to \begin{bmatrix}1&0&0&-1&0&1\\0&1&0&-2&1&1\\0&0&1&2&0&-1\end{bmatrix}$$

Thus $S^{-1} = \begin{pmatrix}-1&0&1\\-2&1&1\\2&0&-1\end{pmatrix}$. Now $A = SDS^{-1}$ with $D = \operatorname{diag}(-2,1,1)$, so we obtain

$$\begin{aligned}
A &= \begin{pmatrix}1&0&1\\0&1&1\\2&0&1\end{pmatrix} \cdot \operatorname{diag}(-2,1,1) \cdot \begin{pmatrix}-1&0&1\\-2&1&1\\2&0&-1\end{pmatrix}\\
&= \begin{pmatrix}-2&0&1\\0&1&1\\-4&0&1\end{pmatrix} \cdot \begin{pmatrix}-1&0&1\\-2&1&1\\2&0&-1\end{pmatrix}\\
&= \begin{pmatrix}4&0&-3\\0&1&0\\6&0&-5\end{pmatrix}
\end{aligned}$$

Problem 6

Suppose you are given eight 6×6 complex matrices whose cube is zero (i.e., $A^3 = 0$). Show that two of the matrices must be similar.

Solution: Since the cube is zero, each matrix has 0 as an eigenvalue and has no other eigenvalues. Also, since the cube is zero, each Jordan block appearing in the Jordan canonical form can be at most 3×3. Thus, we must determine the possible number of Jordan blocks for each size: 1×1, 2×2, and 3×3. The possibilities for the number and sizes of the Jordan blocks in the Jordan canonical form for each matrix are as follows. The list is organized according to the largest block, and then in decreasing number of that block.

(1) two 3×3 Jordan blocks;

(2) one 3×3 Jordan block, one 2×2 block, and one 1×1 block;

(3) one 3×3 Jordan block and three 1×1 blocks;

(4) three 2×2 Jordan blocks;

(5) two 2×2 Jordan blocks and two 1×1 blocks;

(6) one 2×2 Jordan blocks and four 1×1 blocks;

(7) six 1×1 blocks.

Since there are only seven possibilities for Jordan canonical forms for these matrices and we have eight matrices, by the Pigeonhole Principle there must be two matrices A and A' that have the same Jordan canonical form. Since matrices that have the same Jordan canonical form are similar, A and A' are similar.

Problem 7

Let V be a real finite-dimensional inner-product space with proper subspaces U and W. Let P_U and P_W be the orthogonal projections onto U and W, respectively.

(a) Prove or give a counterexample: $P_U P_W = P_W P_U = P_{U \cap W}$.

(b) Prove that trace$(P_U) = \dim U$.

(c) Let $V = \mathbb{C}^n$, and let dim U=1. Write down $\mathcal{M}(P_U)$ with respect to the standard basis for V.

Solution:

(a) A counterexample exists if U and W are disjoint subspaces. Clearly $P_{U \cap W} = 0$. However, given $v \in V$, $P_U P_W v$ is equal to $P_U w = u$, where $w \in W = u + u^{\perp}$. Similarly, $P_W P_U v = \tilde{w}$. This is only true if $u = \tilde{w} = 0$, which means that one space is a subset of the orthogonal complement of the other space. This is not true in general. For example, take two lines through the origin in $\mathbb{R}^2$ that are not perpindicular.

The statement is true if we specify that $U \cap W$ is non-trivial. We may write any vector v in V as $y + z$, with y in $U \cap W$ and z in its orthogonal complement. Clearly $P_{U \cap W} v = y$, while $P_U P_W (y + z) = y + P_U P_W z \to y + P_U \tilde{w}$, where $\tilde{w}$ is in $W \cap U^{\perp}$. But P_U on an object in $U^{\perp}$ is zero, so $P_U P_W v = y$. A symmetric argument holds for $P_W P_U$.

(b) The trace of an operator is the sum of its eigenvalues. All u in U are eigenvectors with eigenvalue 1(as $P_U u = u$). All (non-zero) vectors in $U^{\perp}$ are eigenvectors with eigenvalue 0. As $V = U \oplus U^{\perp}$, the trace of P_U is equal to the number of vectors in the basis for U, i.e. the dimension of U.

(c) If U is one dimensional then there exists a basis vector u than spans U. The orthogonal projection of a vector v onto U is $\frac{<u,v>u}{<u,u>}$. The corresponding matrix is $\frac{uu^{\top}}{u^{\top}u}$.

Problem 8

Let T be an operator on a finite-dimensional complex valued vector space V. Suppose that there exists a $v \in V$ such that $\{v, Tv, T^2v, \ldots T^{n-1}v\}$ is a basis for V.

(a) Prove that $\mathcal{M}(T)$ can be written with respect to this basis such that the matrix is zero below the principal subdiagonal (i.e., that the matrix is upper Hessenberg).

(b) Let T be diagonalizable. Prove that T is invertible.

(c) If T is not invertible, prove that the range of T is invariant under T.

Solution:

(a) $\mathcal{M}(T)$ for the given basis: given ith basis element $b_i = T^{i-1}v$, the corresponding ith column in $\mathcal{M}(T)$ is $T(b_i) = T^i v$. For $i < n$ this is b_{i+1}, so the correspoding ith column is 0 in all entries except the $(i+1)$st row. We do not know what $T^n v$ is, but as b is a basis we can write the last column as a linear combination of the basis functions with coefficients a_i. The resulting matrix is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & a_1 \\ 1 & 0 & 0 & \ldots & 0 & a_2 \\ 0 & 1 & 0 & \ldots & 0 & a_3 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_n \end{pmatrix}$$

which is upper Hessenberg.

(b) As T is diagonalizable, $V = \text{range } T + \text{null } T$. By the above basis, dim range T is at least $n-1$ as $T(v), T(Tv), \ldots T(T^{n-2}v)$ are linearly independent and in range T. If v is in range T, we are done, as T is surjective. If v is not in range T, it is in null T by the direct sum decomposition. But then Tv would be zero, a contradiction.

(c) This result is trivially true. Let w be in range T. Then $w = Tv$, and $z = Tw = T^2v = T(Tv)$, so z is in the range of T. Thus range T is invariant under T.