# University of Colorado Denver
# Department of Mathematical and Statistical Sciences
# Applied Linear Algebra Ph.D. Preliminary Exam
# January 13, 2017

Name: ________________________________________________

**Exam Rules:**

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . . ).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

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DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

**Applied Linear Algebra Preliminary Exam Committee:**
Steffen Borgwardt, Varis Carey, and Stephen Hartke (Chair).

## Problem 1

(a) Consider the set $V = \mathbb{R}^2$ with the usual addition $+$ and the "scalar multiplication" $* : \mathbb{R} \times V \to V$ given by

$$a * (x, y)^T := (a \cdot x, |a| \cdot y)^T \qquad \text{for } a \in \mathbb{R}, \text{ and } (x, y)^T \in V.$$

Prove that $V$ with $+$ (as addition) and $*$ as multiplication satisfies all conditions for a vector space but one. State which condition fails and give an example.

(b) Let $V$ and $W$ be vector spaces over $\mathbb{C}$ and let $f : V \to W$ be linear. Prove that if $U \subseteq V$ is a vector subspace, then the image

$$f(U) := \{f(u) : u \in U\} \subseteq W$$

also is a vector subspace.

## Problem 2

For each $a \in \mathbb{R}$ let the vector subspace $U_a \subseteq \mathbb{R}^4$ be the set of solutions for the linear equation

$$x_1 + x_2 + ax_4 = 0.$$

(a) For each $a, b \in \mathbb{R}$ with $a \neq b$, compute the intersection $U_a \cap U_b$ of the subspaces corresponding to $a$ and $b$, and prove that the intersection does not depend on the choice of $a, b \in \mathbb{R}$ if $a \neq b$.

(b) For which $a, b \in \mathbb{R}$ is $U_a \cup U_b$ a subspace? Prove your claim and give explicit counterexamples for any negative cases.

## Problem 3

Prove that the largest singular value of a linear transformation $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is equal to

$$\max_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \frac{\langle y, Ax \rangle}{\|x\| \, \|y\|}.$$

## Problem 4

(a) Let three bases $A, B, C$ in $\mathbb{R}^3$ be given, as well as the basis transformation matrices

$$S_{A,B} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 0 \\ 0 & 2 & 7 \end{pmatrix} \quad \text{and} \quad S_{A,C} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

Compute the basis transformation matrix $S_{C,B}$.

*Note:* $S_{A,B}$ is the representation matrix of the identity $id : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto x$, where the coordinates of $x$ with respect to $B$ are given before the mapping, and the coordinates with respect to $A$ are given after the mapping.

(b) Let $B = \{b_1, b_2, b_3\} \subset V := \mathbb{R}^3$ with $b_1 = (1, 1, 1)^T$, $b_2 = (2, 0, -1)^T$, $b_3 = (2, 3, 1)^T$ be a basis of $\mathbb{R}^3$ and

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

i) Prove that $\varphi_A : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$ is a vector space isomorphism.

ii) Compute a basis $C$ of $V$, such that $D_{B,C}(\varphi) = I_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $D_{B,C}(\varphi)$ is the representation matrix of $\varphi$. ($D_{B,C}(\varphi)$ takes coordinates with respect to $B$ before the mapping and gives coordinates with respect to $C$ after the mapping.)

## Problem 5

(a) Let $A \in \mathbb{C}^{n \times n}$ be a square matrix and let $\tau \in \mathbb{C}$ be a scalar. For which $\mu \in \mathbb{C}$ is the following statement true? Prove your claim.

Let $\lambda \in \mathbb{C}$. Then $\lambda$ is an eigenvalue of $A$ if and only if $\mu$ is an eigenvalue of $A + \tau \cdot I_n$, where $I_n$ is the $n \times n$ identity matrix.

(b) For a real number $a$, let

$$A = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & 0 \\ 0 & -a & a \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

Compute $A^{101}$.

## Problem 6

a) Let $A \in \mathbb{C}^{n\times n}$ be a *nilpotent* matrix, which means that there is a $k \in \mathbb{N}$, such that $A^k = 0$. Compute all eigenvalues of $A$.

b) Let

$$A = \begin{pmatrix} 12 & 20 & -2 & -10 \\ -3 & -4 & 1 & -1 \\ 9 & 14 & -2 & -4 \\ 3 & 6 & 0 & -6 \end{pmatrix}.$$

Compute $A^2$ and $A^3$, all corresponding eigenvalues and their algebraic and geometric multiplicities. Further, state which eigenspaces (of both matrices) are subsets of each other.

## Problem 7

Let $I$ be the identity operator on $\mathbb{C}^2$. Prove or disprove that $I$ has infinitely many self-adjoint square roots.

## Problem 8

Let $A$ be a real $n \times n$ matrix. A *real cube root* of $A$ is a real $n \times n$ matrix $B$ satisfying $B^3 = A$.

(a) Show that if $A$ is symmetric then it has a real cube root.

(b) Find (with proof) a real $3 \times 3$ matrix which does not have a real cube root.