**University of Colorado Denver**
**Department of Mathematical and Statistical Sciences**
**Applied Linear Algebra Ph.D. Preliminary Exam**
**June 13, 2014**

Name: ______________________________________________

**Exam Rules**:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

$\boxed{\text{Good luck!}}$

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| 1. | ________ | 5. | ________ |
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Total ________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

**Applied Linear Algebra Preliminary Exam Committee:**
Joshua French (Chair), Julien Langou, Anatolii Puhalskii.

1. Assume the following general definition for a real positive semidefinite matrix: an $n \times n$ real matrix $A$ is said to be *positive semidefinite* if and only if, for all vector $x$ in $\mathbb{R}^n$, $x^T Ax \geq 0$. In particular, this definition allows real matrices which are *not symmetric* to be *positive semidefinite*.

   (a) Prove that if $A$ and $B$ are real symmetric positive semidefinite matrices and matrix $A$ is nonsingular, then $AB$ has *only* real nonnegative eigenvalues. (10 pts)

   (b) Provide a counterexample showing that the requirement that the matrices are symmetric cannot be dropped. (10 pts)

2. (a) Suppose $A$ and $B$ are real-valued symmetric $n \times n$ matrices. Show that $|\text{trace}\ (AB)| \leq \sqrt{\text{trace}\ (A^2)}\sqrt{\text{trace}\ (B^2)}$. What are the conditions for equality to hold? (10 pts)

(b) Let $A$ be a real $m \times n$ matrix. Show that

$$\sqrt{\text{trace}(AA^T)} \leq \text{trace}\left(\sqrt{AA^T}\right).$$

When does equality hold ? (10 pts)

3. Let

$$\begin{aligned} f : \mathcal{M}_n(\mathbb{R}) &\longrightarrow \mathcal{M}_n(\mathbb{R}) \\ A &\longmapsto A^T \end{aligned}$$

(a) What are the eigenvalues of $f$? (10 pts)

(b) Is $f$ diagonalizable? If yes, give a basis of eigenvectors. If no, give as many linearly independent eigenvectors as possible. (10 pts)

4. Define the $n \times n$ matrix

$$A_n = \begin{bmatrix} a+b & b & b & \dots & b & b \\ a & a+b & b & \ddots & b & b \\ a & a & a+b & \ddots & b & b \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a & a & a & \ddots & a+b & b \\ a & a & a & \dots & a & a+b \end{bmatrix}$$

(a) Compute $D_n = \det(A_n)$. (10 pts)

(b) Give the value of $D_n$ for $n = 10$, $a = 2$, and $b = -1$. (10 pts)

5. Suppose that $u$ and $v$ are vectors in a real inner product space $V$.

   (a) Prove that

$$(||u|| + ||v||)\frac{\langle u, v\rangle}{||u||\,||v||} \le ||u+v||. \quad \text{(10 pts)}$$

   (b) Prove or disprove the following identity:

$$(||u|| + ||v||)\frac{|\langle u, v\rangle|}{||u||\,||v||} \le ||u+v||. \quad \text{(10 pts)}$$

6. Let $V$ be a vector space. Let $f \in \mathcal{L}(V)$. Let $p$ be a projection (so $p \in \mathcal{L}(V)$ and is such that $p^2 = p$). Prove that

$$\operatorname{Null}(f \circ p) = \operatorname{Null}(p) \oplus (\operatorname{Null}(f) \cap \operatorname{Range}(p)) . \quad \text{(20 pts)}$$

7. (a) Let $n \in \mathbb{N} \backslash \{0, 1\}$ (so $n \geq 2$) and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\text{rank}(A) = 1$. Prove that $A$ is diagonalizable if and only if $\text{trace}(A) \neq 0$. (10 pts)

   (b) Let $a_1, \ldots a_n \in \mathbb{C} \backslash \{0\}$, (so the $a_i$'s are nonzero complex numbers,) and $A$ such that $A = \left(\frac{a_i}{a_j}\right)_{1 \leq i,j \leq n}$. (This means that the entry $(i, j)$ of $A$ is $\frac{a_i}{a_j}$.) Show that $A$ is diagonalizable. Give a basis of eigenvectors (with the associated eigenvalues) for $A$. (10 pts)