University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 14, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Find the least squares solution of $Ax = b$ where

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}.$$

2. Let $\mathbb{F}$ be a field. Let $\mathcal{P}_1$ denote the standard vector space of polynomials $f(t)$ with coefficients in the field $\mathbb{F}$ and having degree at most 1. Let $\mathcal{S} = \{1, t\}$ be the standard ordered basis of $\mathcal{P}_1$.

   (a) Define $T \in \mathcal{L}(\mathcal{P}_1)$ by

   $$T : p(t) = a + bt \mapsto q(t) = 5a - 2b + (4a - b)t.$$

   Construct the matrix $A = [T]_{\mathcal{S}}$ that represents T with respect to the basis $\mathcal{S}$. Is there an ordered basis $\mathcal{B}$ for $\mathcal{P}_1$ such that $[T]_{\mathcal{B}}$ is diagonal? If so, give such a basis and the corresponding matrix representation. If not, explain why not.

   (b) Replace T of part (a) by $S \in \mathcal{L}(\mathcal{P}_1)$ defined by

   $$S : p(t) = a + bt \mapsto q(t) = -a + b - bt,$$

   and repeat question (a).

3. Let A be a real matrix. A *generalized inverse* of a matrix A is any matrix G such that $AGA = A$. Prove each of the following:

   (a) If A is invertible, the unique generalized inverse of A is A^{-1}.

   (b) If G is a generalized inverse of $(X^T X)$, then

   $$XGX^T X = X \ .$$

   (c) For any real symmetric matrix A, there exists a generalized inverse of A.

4. Let A be a real symmetric n-by-n matrix which is not just a scalar multiple of the identity matrix. Let $f(x) = (x - 1)(x + 6)^3$ and suppose that $f(A) = 0$ and the trace of A is 0.

   (a) Determine the minimal polynomial of A.

   (b) Determine the trace of A^2 as a function of n.

   (c) Show that n is a multiple of 7.

   (d) Determine the characteristic polynomial of A as a function of n.

5. Let U and W be subspaces of the finite-dimensional inner product space V.

(a) Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.

(b) Prove that

$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

6. Let B be an n-by-n Hermitian matrix. Then B has real eigenvalues which we may order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $\overline{0} \neq \mathbf{x} \in \mathbb{C}^n$, and using the usual 2-norm $\|\mathbf{x}\| = \|\mathbf{x}\|_2$, define the Rayleigh Quotient $\rho_B(\mathbf{x})$ for B by

$$\rho_B(\mathbf{x}) = \frac{\langle B\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\mathbf{x}^* B\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Prove the following:

(i) If B is an n-by-n Hermitian with eigenvalues as above, prove that $\lambda_1 = \max\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathbb{C}^n \text{ and } \|\mathbf{x}\| = 1\}$.

(ii) Let A be any $n \times n$ complex matrix with largest singular value σ_1. If $\|A\|_2 = \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{C}^n \text{ and } \|\mathbf{x}\| = 1\}$, show that

$$\|A\|_2 = \sigma_1.$$

7. Let T be a normal operator on a finite-dimensional complex inner product space V.

   (a) Prove that T is self-adjoint if and only if its eigenvalues are all real.

   (b) Prove that T is positive (i.e., positive semidefinite) if and only if all its eigenvalues are nonnegative.

8. (a) (Frobenius inequality) If A, B, and C are rectangular matrices such that the product ABC is defined, then

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(B) + \operatorname{rank}(ABC)$$

(b) In particular, prove that

$$\operatorname{rank}(AB) \leq \min\left\{\operatorname{rank}(A), \operatorname{rank}(B)\right\}.$$