University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

16 January 2009, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

On this exam V is a finite dimensional vector space over the field F, where either $F = \mathcal{C}$, the field of complex numbers, or $F = \mathcal{R}$, the field of real numbers. Also, F^n denotes the vector space of column vectors with n entries from F, as usual. For $T \in \mathcal{L}(V)$, the *image* (sometimes called the *range*) of T is denoted $\mathrm{Im}(T)$.

1. Suppose that $P \in \mathcal{L}(V)$ (the vector space of linear maps from V to itself) and that $P^2 = P$.
   (a) (6 points) Determine all possible eigenvalues of P.
   (b) (10 points) Prove that $V = \mathrm{null}(P) \oplus \mathrm{Im}(P)$.
   (c) (4 points) Is it necessary that all possible eigenvalues found in part (a) actually must occur? Prove that your answer is correct.

2. Define $T \in \mathcal{L}(F^n)$ by $T\colon (w_1, w_2, w_3, w_4)^T \mapsto (0, w_2 + w_4, w_3, w_4)^T$.
   (a) (8 points) Determine the minimal polynomial of T.
   (b) (6 points) Determine the characteristic polynomial of T.
   (c) (6 points) Determine the Jordan form of T.

3. Let T be a normal operator on a complex inner product space V of dimension n.
   (a) (10 points) If $T(v) = \lambda v$ with $\mathbf{0} \neq v \in V$, show that v is an eigenvector of the adjoint T^* with associated eigenvalue $\overline{\lambda}$.
   (b) (10 points) Show that T^* is a polynomial in T.

4. Let A and B be $n \times n$ Hermitian matrices over $\mathcal{C}$.
   (a) (10 points) If A is positive definite, show that there exists an invertible matrix P such that $P^*AP = I$ and P^*BP is diagonal.
   (b) (10 points) If A is positive definite and B is positive semidefinite, show that
   $$\det(A + B) \geq \det(A).$$

5. Let $\|\cdot\|_\infty\colon \mathcal{C}^n \to \mathcal{R}$ be defined by
   $$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$
   (a) (8 points) Prove that $\|\cdot\|_\infty$ is a norm.
   (b) (12 points) A norm $\|\cdot\|$ is said to be derived from an inner product if there is an inner product $\langle\cdot,\cdot\rangle$ such that $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$ for all $\mathbf{x} \in \mathcal{C}^n$. Show that $\|\cdot\|_\infty$ cannot be derived from an inner product.

6. Suppose that $F = \mathcal{C}$ and that S, $T \in \mathcal{L}(V)$ satisfy $ST = TS$. Prove each of the following:
   (a) (4 points) If λ is an eigenvalue of S, then the eigenspace
   $$V_\lambda = \{\mathbf{x} \in V | S\mathbf{x} = \lambda\mathbf{x}\}$$
   is invariant under T.

(b) (4 points) S and T have at least one common eigenvector (not necessarily belonging to the same eigenvalue).

(c) (12 points) There is a basis $\mathcal{B}$ of V such that the matrix representations of S and T are both upper triangular.

7. Let $F = \mathcal{C}$ and suppose that $T \in \mathcal{L}(V)$.

(a) (10 points) Prove that the dimension of $\mathrm{Im}(T)$ equals the number of nonzero singular values of T.

(b) (10 points) Suppose that $T \in \mathcal{L}(V)$ is positive semidefinite. Prove that T is invertible if and only if $\langle T(\mathbf{x}), \mathbf{x} \rangle > 0$ for every $\mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0}$.

8. Let N be a real $n \times n$ matrix of rank $n - m$ and nullity m. Let L be an $m \times n$ matrix whose rows form a basis of the left null space of N, and let R be an $n \times m$ matrix whose columns form a basis of the right null space of N. Put $Z = L^T R^T$. Finally, put $M = N + Z$.

(a) (2 points) For $\mathbf{x} \in \mathcal{R}^n$, show that $N^T \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = L^T \mathbf{y}$ for some $\mathbf{y} \in \mathcal{R}^m$.

(b) (2 points) For $\mathbf{x} \in \mathcal{R}^n$, show that $N\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = R\mathbf{y}$ for some $\mathbf{y} \in \mathcal{R}^m$.

(c) (4 points) Show that Z is an $n \times n$ matrix with rank m for which $N^T Z = \mathbf{0}$, $NZ^T = \mathbf{0}$ and $MM^T = NN^T + ZZ^T$.

(d) (12 points) Show that the eigenvalues of MM^T are precisely the positive eigenvalues of NN^T and the positive eigenvalues of ZZ^T, and conclude that MM^T is nonsingular.