University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
Aug. 13, 2024

Student Number: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete six problems. You are allowed to take a break of up to 45 minutes; please start this break not earlier than 90 minutes and not later than 150 minutes into the exam.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper and leave at least a half-inch margin.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$ are the vector spaces of n-tuples and $m \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1.

Let $T : V \to W$ and $S : W \to X$ be linear transformations of finite-dimensional real vector spaces. Prove that

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T) \leq \min\{\operatorname{rank}(T), \operatorname{rank}(S)\}.$$

Solution Recall $\operatorname{rank}(T) = \dim(\operatorname{range}(T))$ and note that $\operatorname{rank}(S \circ T) = \operatorname{rank}(T) - \dim(\operatorname{range}(T) \cap \operatorname{null}(S))$. By rank theorem and this observation,

$$\begin{aligned}\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) &= \operatorname{rank}(T) + \operatorname{rank}(S) - \dim(\operatorname{null}(S)) - \operatorname{rank}(S) = \\ &= \operatorname{rank}(T) - \dim(\operatorname{null}(S)) = \operatorname{rank}(S \circ T) + \dim(\operatorname{range}(T) \cap \operatorname{null}(S)) - \dim(\operatorname{null}(S))\end{aligned}$$

and due to

$$\dim(\operatorname{range}(T) \cap \operatorname{null}(S)) \leq \dim(\operatorname{null}(S))$$

we obtain the left-hand inequality

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T).$$

For the right-hand inequality, we observe

$$\operatorname{rank}(S \circ T) = \dim(S(T(V))) \leq \dim(T(V)) = \operatorname{rank}(T)$$

and

$$\operatorname{rank}(S \circ T) = \dim(S(T(V))) \leq \dim(S(W)) = \operatorname{rank}(S).$$

Problem 2.

1. (10 points) Let V be a finite-dimensional inner product space. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

2. (10 points) Let V and W be two finite-dimensional inner product spaces. Let $T \in \mathcal{L}(V, W)$. Prove that

   (a) T is injective if and only if T^* is surjective;

   (b) T is surjective if and only if T^* is injective.

Solution

1. First suppose U is invariant under T. To prove that $U^{\perp}$ is invariant under T^*, let $v \in U^{\perp}$. We need to show that $T^*v \in U^{\perp}$. But

   $$\langle u, T^*v\rangle = \langle Tu, v\rangle = 0$$

   for every $u \in U$ (because if $u \in U$, then $Tu \in U$ and hence Tu is orthogonal to v, an element of $U^{\perp}$). Thus $T^*v \in U^{\perp}$. Hence $U^{\perp}$ is invariant under T^*, as desired.

   To prove the other direction, now suppose $U^{\perp}$ is invariant under T^*. Then by the first direction, we know that $\left(U^{\perp}\right)^{\perp}$ is invariant under $(T^*)^*$. But $\left(U^{\perp}\right)^{\perp} = U$ (by 6.51) and $(T^*)^* = T$, so U is invariant under T, completing the proof.

2. First we prove (a):

   $$\begin{aligned} T \text{ is injective } &\Leftrightarrow \text{null}(T) = \{0\} \\ &\Leftrightarrow (\text{range}(T^*))^{\perp} = \{0\} \\ &\Leftrightarrow \text{range}(T^*) = V \\ &\Leftrightarrow T^* \text{ is surjective} \end{aligned}$$

   where the second line comes from 7.7(c). Now that (a) has been proved, (b) follows immediately by replacing T with T^* in (a).

Problem 3.

1. (10 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric and let $\text{tr}(T^2) = 0$. Show that $T = 0$.
2. (7 points) Give $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is normal, $\text{tr}(T^2) = 0$ and T is not 0.
3. (3 points) Give $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not normal, $\text{tr}(T^2) = 0$ and T is not 0.

Solution

1. By the spectral theorem, $T = ODO^*$ where O is orthogonal and D is a diagonal matrix with real entries $\lambda_1, \ldots, \lambda_n$. Observe that $T^2 = (ODO^*)(ODO^*) = ODDO^* = OD^2O^*$. Now

   $$0 = \text{trace}(T^2) = \text{trace}(OD^2O^*) = \text{trace}(O^*OD^2) = \text{trace}(D^2) = \lambda_1^2 + \cdots + \lambda_n^2.$$

   As $\lambda_i \in \mathbb{R}$ and $\lambda_i^2 \geq 0$ for all $i \in \{1, \ldots, n\}$, $\lambda_1^2 + \cdots + \lambda_n^2 = 0$ implies $\lambda_i = 0$ for all $i \in \{1, \ldots, n\}$.

2. Consider a transformation T represented by matrix

$$A = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix},$$

which is not symmetric but readily verified to be normal, i.e. $AA^* = A^*A$.

The characteristic polynomial is $(2-\lambda)^2 + 4 = \lambda^2 - 4\lambda + 8$, which gives eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(4 \pm \sqrt{16-32}) = 2 \pm \sqrt{-4} = 2 \pm 2i.$$

By the spectral theorem, there exists an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A and corresponding diagonal matrix

$$D = \begin{pmatrix} 2+2i & 0 \\ 0 & 2-2i \end{pmatrix}.$$

Observe that

$$\operatorname{trace}(T^2) = \operatorname{trace}(D^2) = (2+2i)^2 + (2-2i)^2 = 8 + 8i^2 = 0,$$

which disproves the claim.

Scaling in the above example and transpose do not matter, so other examples would be

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

3. An "easy" matrix would be for example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

Then A is not zero, but A^2 is zero, so $\operatorname{trace}(A^2) = 0$. This matrix is not normal.

Another example would be

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Problem 4.

1. Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix

$$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

prove there does not exist a real-valued matrix B such that BAB^{-1} is a diagonal matrix.

2. Let $q \neq 0 \in \mathbb{R}$. For the matrix

$$A = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix},$$

prove there does not exist a real-valued matrix B such that BAB^{-1} is a diagonal matrix.

Solution

1. We have

$$\det(A - \lambda I) = \det \begin{pmatrix} \cos(t) - \lambda & \sin(t) \\ -\sin(t) & \cos(t) - \lambda \end{pmatrix} = \lambda^2 - 2\lambda\cos(t) + 1,$$

which gives $\lambda_{1,2} = \cos(t) \pm \sqrt{\cos^2(t) - 1}$. As t is not an integer multiple of π, $\cos^2(t) - 1 < 0$ and $\lambda_{1,2} = a \pm bi$ for some a and some $b \neq 0$, i.e., $\lambda_{1,2} \notin \mathbb{R}$. Thus, eigenvectors are not real either, which implies that there cannot exist a real-valued B such that BAB^{-1} is diagonal.

2. As A is an upper-triangular matrix with diagonal entries 1, it only has a single eigenvalue $\lambda = \lambda_{1,2} = 1$. The corresponding eigenspace is

$$\operatorname{null} \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Observe that the geometric multiplicity strictly exceeds the algebraic multiplicity. This implies that there does not exist a basis of $\mathbb{R}^2$ consisting of eigenvectors of A, and in turn that there cannot exist a real-valued B such that BAB^{-1} is diagonal.

Part II. Work **two** of problems 5 through 8.

Problem 5. Let m be a positive integer. Suppose $p_1, \ldots, p_m$ are polynomials in the space $\mathcal{P}_m(\mathbb{R})$ (of polynomials over $\mathbb{R}$ with degree at most m) such that $p_j(2) = 0$ and $p_j(5) = 0$ for each j. Prove that the set $\{p_1, \ldots, p_m\}$ is linearly dependent in $\mathcal{P}_m(\mathbb{R})$.

Solution Method 1:
Let S be the subset of $\mathcal{P}_m(\mathbb{R})$ consisting of the polynomials $q(x)$ where $q(2) = 0$ and $q(5) = 0$. We verify that S is a subspace of $\mathcal{P}_m(\mathbb{R})$:

1. *S is nonempty.* The zero polynomial is in S.
2. *S is closed under addition.* Let $q, b \in S$. Then $(q+b)(2) = q(2)+b(2) = 0+0 = 0$, and $(q+b)(5) = q(5)+b(5) = 0+0 = 0$. Hence, $q+b \in S$.
3. *S is closed under scalar multiplication.* Let $q \in S$ and $c \in \mathbb{R}$. Then $(cq)(2) = c(q(2)) = c0 = 0$, and $(cq)(5) = c(q(5)) = c0 = 0$. Hence, $cq \in S$.

Let $f(x) = x - 5$ and $g(x) = x - 2$. Note that $\{f, g\}$ is linearly independent in $\mathcal{P}_m(\mathbb{R})$ due to the difference in constant terms. Let $T = \operatorname{span}\{f, g\}$. Since $\{f, g\}$ is linearly independent, $\dim T = 2$.

Suppose that $c_1 f(x) + c_2 g(x) = q(x) \in S$. Then

$$0 = q(2) = c_1(2-5) + c_2(2-2) = -3c_1.$$

Hence $c_1 = 0$. Similarly, $c_2 = 0$. Therefore, $S \cap T = \{0\}$.

Thus, the direct sum $S \oplus T$ is a subspace of $\mathcal{P}_m(\mathbb{R})$, and we have

$$\begin{aligned} \dim(S \oplus T) &\leq \dim \mathcal{P}_m(\mathbb{R}) \\ \dim S + \dim T &\leq m+1 \\ \dim S &\leq m-1. \end{aligned}$$

Since the set $\{p_1, \ldots, p_m\}$ has m elements lying in a vector space of dimension at most $m-1$, the set is linearly dependent.

Method 2:
Let i be an integer between 1 and m. Since $p_i(x)$ is a polynomial in $\mathcal{P}_m(\mathbb{R})$ (polynomial of degree less than or equal to m) such that $p_i(2) = 0$ and $p_i(5) = 0$, $p_i(x)$ is of the form

$$p_i(x) = (x-2)(x-5)q_i(x),$$

where $q_i(x)$ is in $\mathcal{P}_{m-2}(\mathbb{R})$ (polynomial of degree less than or equal to $m-2$). Now since $q_1, \ldots, q_m$ are m polynomials in $\mathcal{P}_{m-2}(\mathbb{R})$ and the dimension of $\mathcal{P}_{m-2}(\mathbb{R})$ is $m-1$, it must be that $\{q_1, \ldots, q_m\}$ is linearly dependent. Since $\{q_1, \ldots, q_m\}$ is linearly dependent, there exists m not-all-zero scalars α_1, ..., α_m, such that $\alpha_1 q_1 + \ldots + \alpha_m q_m = 0$. Multiplying by $(x-2)(x-5)$, we obtain that $\alpha_1 p_1 + \ldots + \alpha_m p_m = 0$, with the α_1, ..., α_m that are not-all-zero. Therefore $\{p_1, \ldots, p_m\}$ is linearly dependent.

Problem 6. For integer $n, m \geq 1$, consider the subset $S(n,m)$ of $\mathbb{R}^{n\times m}$ consisting of all matrices for which the sum of all even-indexed column vectors equals the sum of all odd-indexed column vectors. That is, $S(n,m)$ consists of all matrices $A \in \mathbb{R}^{n\times m}$ whose column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ satisfy

$$\sum_{i \text{ even}} \mathbf{a}_i = \sum_{i \text{ odd}} \mathbf{a}_i.$$

1. (4 pts) Show that $S(n,m)$ is a vector subspace of $\mathbb{R}^{n\times m}$.
2. (8 pts) Show that for any $A \in \mathbb{R}^{m\times n}$ and $B \in S(n,m)$, their product AB is in $S(m,m)$.
3. (8 pts) Let $A \in S(n,m)$. What is the least eigenvalue of $I_m + A^T A$?

Solution

1. To show that $S = S(n,m)$ is a subspace of $\mathbb{R}^{n\times m}$ we need only show that S is S is nonempty; S is closed under vector addition; and S is closed under scalar multiplication.

   (a) *S is nonempty.* Note that the $m \times n$ zero matrix is in S.

   (b) *S is closed under vector addition.* Let $A, B \in S$, with corresponding column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$. Since $A, B \in S$, we have that

   $$\sum_{i \text{ even}} \mathbf{a}_i = \sum_{i \text{ odd}} \mathbf{a}_i, \qquad \sum_{i \text{ even}} \mathbf{b}_i = \sum_{i \text{ odd}} \mathbf{b}_i.$$

   Adding the two equations together and rearranging, we obtain

   $$\sum_{i \text{ even}} (\mathbf{a}_i + \mathbf{b}_i) = \sum_{i \text{ odd}} (\mathbf{a}_i + \mathbf{b}_i).$$

   Thus, $A + B \in S$.

   (c) *S is closed under scalar multiplication.* Let $A \in S$, with corresponding column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$, and let $c \in R$ be a scalar. Since $A \in S$, we have that

   $$\sum_{i \text{ even}} \mathbf{a}_i = \sum_{i \text{ odd}} \mathbf{a}_i.$$

   Multiplying both sides of the equation by c and rearranging, we obtain

   $$\sum_{i \text{ even}} c\mathbf{a}_i = \sum_{i \text{ odd}} c\mathbf{a}_i.$$

   Thus, $cA \in S$. Hence, S is a subspace of $\mathbb{R}^{n\times m}$.

2. Let $\mathbf{v} \in \mathbb{R}^m$ where the even coordinates are 1 and the odd coordinates are -1; that is

$$\mathbf{v} = (1, -1, 1, -1, \dots).$$

Observe that $B \in \mathbb{R}^{n\times m}$ is in $S(n,m)$ if and only if $B\mathbf{v} = \mathbf{0}$. (Note this observation can be used as an alternate approach to the first part.) Let $A \in \mathbb{R}^{m\times n}$ and $B \in S(n,m)$. Then

$$(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0} = \mathbf{0}.$$

Thus, $AB \in S(m,m)$.

3. From the second part, $A^TA \in S(m,m)$. Since A satisfies a linear dependence among its columns (alternatively, that it has a nontrivial nullspace), the rank of A is less than m. Thus the rank of A^TA is also less than m, and hence A^TA is not invertible. Since A^TA is symmetric and real, by the spectral theorem it has nonnegative real roots. Thus, the least eigenvalue of A^TA is 0. The eigenvalues of $I_m + A^TA$ are the eigenvalues of A^TA shifted by 1. Thus, the least eigenvalue of $I_m + A^TA$ is 1.

Problem 7.

1. (12 points)

Let $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$.

Find a least-squares solution x to $\min_x \|Ax - b\|_2$.

2. (8 points) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Check that x is a least-squares solution to $\min_x \|Ax - b\|_2$.

Solution

1. We use the method of normal equations.

$$A^TA = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

We now solve $(A^TA)x = (A^Tb)$, which is

$$\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

It might be a good idea to simplify by 6:

$$\begin{pmatrix} 1 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We now solve and we find

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

A solution to the least squares problem is

$$x^* = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Note: Because A^TA is invertible, (or equivalently, because A is full column rank,) this solution x is unique.

Note: We can check that x^* is a solution to the least squares problem by computing the following:

$$b - Ax^* = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$$

$$A^T(b - Ax^*) = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2.

$$
\begin{aligned}
Ax &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (1)(-1) + (1)(1) + (1)(1) + (1)(1) \\ (-1)(-1) + (1)(1) + (-1)(1) + (1)(1) \\ (0)(-1) + (-1)(1) + (1)(1) + (1)(1) \\ (0)(-1) + (1)(1) + (0)(1) + (-1)(1) \\ (1)(-1) + (1)(1) + (-1)(1) + (1)(1) \end{pmatrix} \\
&= \begin{pmatrix} -1 + 1 + 1 + 1 \\ 1 + 1 - 1 + 1 \\ 0 - 1 + 1 + 1 \\ 0 + 1 + 0 - 1 \\ -1 + 1 - 1 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
b - Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}
$$

$$
\begin{aligned}
A^T(b - Ax) &= \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (1)(-1) + (-1)(0) + (0)(2) + (0)(2) + (1)(1) \\ (1)(-1) + (1)(0) + (-1)(2) + (1)(2) + (1)(1) \\ (1)(-1) + (-1)(0) + (1)(2) + (0)(2) + (-1)(1) \\ (1)(-1) + (1)(0) + (1)(2) + (-1)(2) + (1)(1) \end{pmatrix} \\
&= \begin{pmatrix} -1 + 0 + 0 + 0 + 1 \\ -1 + 0 - 2 + 2 + 1 \\ -1 + 0 + 2 + 0 - 1 \\ -1 + 0 + 2 - 2 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Problem 8. $\mathcal{M}_n(\mathbb{C})$ is the set of complex n-by-n matrices.

1. (7 points) Show that if C in $\mathcal{M}_n(\mathbb{C})$ and $x^H Cx = 0$ for all x in $\mathbb{C}^n$, then $C = 0$. While this is not needed to prove the result, you may assume that C is Hermitian.

2. (6 points) Show that for any A in $\mathcal{M}_n(\mathbb{C})$ there are Hermitian matrices B and C for which $A = B + iC$.

3. (7 points) Let A in $\mathcal{M}_n(\mathbb{C})$. Show that if $x^H Ax$ is real for all x in $\mathbb{C}^n$, then A is Hermitian.

<u>Hint 1:</u> *Part 1 and Part 2 are independent. Part 3 can be efficiently done using results from Part 1 and from Part 2.*

<u>Hint 2:</u> *Part 1 can be proven for a general matrix C in $\mathcal{M}_n(\mathbb{R})$, however you may assume that C is Hermitian to prove the result. This might lead to an easier proof. Note: Part 3 only needs the result of Part 1 in the special case when C is Hermitian. Part 3 does not need the general result of Part 1.*

Solution

1. Some comments before starting:

   (a) This result is Theorem 7.14 in Axler. The proof in Axler relies on the identity, for any vectors u and w, and any linear operator T:

   $$\begin{aligned}\langle Tu, w\rangle &= \frac{1}{4}(\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle \\ &+i\langle T(u+iw), u+iw\rangle - i\langle T(u-iw), u-iw\rangle).\end{aligned}$$

   We propose a different proof below.

   (b) We note that a similar result in $\mathbb{R}$ is not correct. It is not true that, if C in $\mathcal{M}_n(\mathbb{R})$ is such that $x^T Cx = 0$ for all x in $\mathbb{R}^n$, then $C = 0$. This would be true if we add the assumption that C is self-adjoint.

   (c) The result is "easier" to prove in $\mathcal{M}_n(\mathbb{C})$ if we assume C to be self-adjoint. (Then we can use the spectral theorem and be done pretty quickly.) This "easier" and "weaker" result is all we will need in Part 3.

   Let C in $\mathcal{M}_n(\mathbb{C})$ such that $x^H Cx = 0$ for all x in $\mathbb{C}^n$.

   Let ℓ be an integer between 1 and n. Let e_ℓ be the vector of all zeros but a 1 in position ℓ. Then $e_\ell^H Ce_\ell = c_{\ell\ell}$ and so we find that we must have

   $$c_{\ell\ell} = 0.$$

   So the diagonal of C has to be zero.

   Let j and k be two distinct integers between 1 and n. Let $f_{j,k}$ the vector of all zeros but a 1 in position j and a 1 in position k. Then $f_{j,k}^H Cf_{j,k} = c_{jj} + c_{jk} + c_{kj} + c_{kk}$, so that $c_{jj} + c_{jk} + c_{kj} + c_{kk} = 0$. Since we proved that the diagonal elements of C are zeros, we get

   $$c_{jk} + c_{kj} = 0. \tag{1}$$

   Let $g_{j,k}$ be the vector of all zeros but a 1 in position j and an i in position k. Then $g_{j,k}^H Cg_{j,k} = c_{jj} + ic_{jk} - ic_{kj} + c_{kk}$, so that $c_{jj} + ic_{jk} - ic_{kj} + c_{kk} = 0$. Since we proved that the diagonal elements of C are zeros, we get

   $$c_{jk} - c_{kj} = 0. \tag{2}$$

   Combining Equations (1) and (2), we find

   $$c_{jk} = 0 \quad \text{and} \quad c_{kj} = 0.$$

   This proves that all off-diagonal entries of C are zeros.

   Therefore

   $$C = 0.$$

2. We will do a proof by construction. We assume that $A = B + iC$ with $B = B^H$ and $C = C^H$, therefore $A^H = B^H - iC^H = B - iC$. So $A = B + iC$ and $A^H = B - iC$, we can solve for B and C and we get

$$B = \frac{1}{2}\left(A + A^H\right) \quad \text{and} \quad C = -\frac{i}{2}\left(A - A^H\right).$$

Note: We can check that this B and this C are Hermitian and that indeed $A = B + iC$.

Note: There is no other solution. So this B and this C are unique.

3. Let A in $\mathcal{M}_n(\mathbb{C})$ such that $x^H A x$ is real for all x in $\mathbb{C}^n$.

From Part 2, we can write $A = B + iC$ where both B and C are Hermitian. Then we have $x^H A x = x^H B x + i x^H C x$. We know that $x^H A x$ is real by assumption. And because B and C are Hermitian, $x^H B x$ and $x^H C x$ are real as well. Therefore it must be that $x^H C x = 0$. By Part 1, either the general result, or the result with C Hermitian (since C is Hermitian in our case), we conclude that $C = 0$. Since $C = 0$, we find that $A = B$ and so A is Hermitian.