University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Aug. 13, 2024

Student Number: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete six problems. You are allowed to take a break of up to 45 minutes; please start this break not earlier than 90 minutes and not later than 150 minutes into the exam.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper and leave at least a half-inch margin.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$ are the vector spaces of n-tuples and $m \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Steffen Borgwardt (Chair), Stephen Hartke, Julien Langou

Part I. Work **all** of problems 1 through 4.

Problem 1.

Let $T : V \to W$ and $S : W \to X$ be linear transformations of finite-dimensional real vector spaces. Prove that

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T) \leq \min\{\operatorname{rank}(T), \operatorname{rank}(S)\}.$$

Problem 2.

1. (10 points) Let V be a finite-dimensional inner product space. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V. Prove that U is invariant under T if and only if $U^{\perp}$ is invariant under T^*.
2. (10 points) Let V and W be two finite-dimensional inner product spaces. Let $T \in \mathcal{L}(V, W)$. Prove that
   (a) T is injective if and only if T^* is surjective;
   (b) T is surjective if and only if T^* is injective.

Problem 3.

1. (10 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric and let $\operatorname{tr}(T^2) = 0$. Show that $T = 0$.
2. (7 points) Give $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is normal, $\operatorname{tr}(T^2) = 0$ and T is not 0.
3. (3 points) Give $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not normal, $\operatorname{tr}(T^2) = 0$ and T is not 0.

Problem 4.

1. Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix
$$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$
prove there does not exist a real-valued matrix B such that BAB^{-1} is a diagonal matrix.

2. Let $q \neq 0 \in \mathbb{R}$. For the matrix
$$A = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix},$$
prove there does not exist a real-valued matrix B such that BAB^{-1} is a diagonal matrix.

Part II. Work **two** of problems 5 through 8.

Problem 5. Let m be a positive integer. Suppose $p_1, \ldots, p_m$ are polynomials in the space $\mathcal{P}_m(\mathbb{R})$ (of polynomials over $\mathbb{R}$ with degree at most m) such that $p_j(2) = 0$ and $p_j(5) = 0$ for each j. Prove that the set $\{p_1, \ldots, p_m\}$ is linearly dependent in $\mathcal{P}_m(\mathbb{R})$.

Problem 6. For integer $n, m \geq 1$, consider the subset $S(n, m)$ of $\mathbb{R}^{n \times m}$ consisting of all matrices for which the sum of all even-indexed column vectors equals the sum of all odd-indexed column vectors. That is, $S(n, m)$ consists of all matrices $A \in \mathbb{R}^{n \times m}$ whose column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ satisfy
$$\sum_{i \text{ even}} \mathbf{a}_i = \sum_{i \text{ odd}} \mathbf{a}_i.$$

1. (4 pts) Show that $S(n, m)$ is a vector subspace of $\mathbb{R}^{n \times m}$.
2. (8 pts) Show that for any $A \in \mathbb{R}^{m \times n}$ and $B \in S(n, m)$, their product AB is in $S(m, m)$.
3. (8 pts) Let $A \in S(n, m)$. What is the least eigenvalue of $I_m + A^T A$?

Problem 7.

1. (12 points)

   Let $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$.

   Find a least-squares solution x to $\min_x \|Ax - b\|_2$.

2. (8 points) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

   Check that x is a least-squares solution to $\min_x \|Ax - b\|_2$.

Problem 8. $\mathcal{M}_n(\mathbb{C})$ is the set of complex n-by-n matrices.

1. (7 points) Show that if C in $\mathcal{M}_n(\mathbb{C})$ and $x^H Cx = 0$ for all x in $\mathbb{C}^n$, then $C = 0$. While this is not needed to prove the result, you may assume that C is Hermitian.

2. (6 points) Show that for any A in $\mathcal{M}_n(\mathbb{C})$ there are Hermitian matrices B and C for which $A = B + iC$.

3. (7 points) Let A in $\mathcal{M}_n(\mathbb{C})$. Show that if $x^H Ax$ is real for all x in $\mathbb{C}^n$, then A is Hermitian.

Hint 1: *Part 1 and Part 2 are independent. Part 3 can be efficiently done using results from Part 1 and from Part 2.*

Hint 2: *Part 1 can be proven for a general matrix C in $\mathcal{M}_n(\mathbb{R})$, however you may assume that C is Hermitian to prove the result. This might lead to an easier proof. Note: Part 3 only needs the result of Part 1 in the special case when C is Hermitian. Part 3 does not need the general result of Part 1.*
