University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
Jan. 12, 2024

Student Number: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete six problems. You are allowed to take a break of up to 45 minutes; please start this break not earlier than 90 minutes and not later than 150 minutes into the exam.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper and leave at least a half-inch margin.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$ are the vector spaces of n-tuples and $m \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1. Let $a \neq 0, b \neq 0 \in \mathbb{R}$ be fixed. The below questions require a case distinction based on values of a and b. Consider the matrix

$$A = \begin{pmatrix} a-b & a+b & a+b & a-b \\ 0 & 0 & a-b & a-b \\ 0 & a-b & 0 & b-a \\ b-a & 0 & 0 & b-a \end{pmatrix}.$$

1. (10 points) Find a basis for null(A).
2. (5 points) Find a basis for range(A).
3. (5 points) Find a basis for the subspace $S = \text{null}(A) \cap \text{range}(A)$.

Solution We answer the questions with a case distinction based on $a = b$ or $a \neq b$. Let e_i denote the i-th unit vector. First, let $a = b$.

1. If $a = b$, then the matrix consists of two (scaled) unit columns of the form $2a \cdot e_1$ in columns 2 and 3, and is zero otherwise. Thus $\text{null}(A) = \text{span}\{e_1, e_2 - e_3, e_4\}$.
2. If $a = b$, the first and fourth vector are zero and vectors 2 and 3 are $2a \cdot e_1$. Thus $\{e_1\}$ is a basis of range(A).
3. For $a = b$, note that $e_1 \in \text{span}\{e_1, e_2 - e_3, e_4\}$. Thus $\{e_1\}$ is a basis of S.

Now, let $a \neq b$.

1. If $a \neq b$, then the matrix can be row-reduced to the form

$$\rightarrow \begin{pmatrix} a-b & a+b & a+b & a-b \\ 0 & a-b & 0 & b-a \\ 0 & 0 & a-b & a-b \\ 0 & a+b & a+b & 0 \end{pmatrix} \rightarrow \begin{pmatrix} a-b & 0 & 0 & a-b \\ 0 & a-b & 0 & b-a \\ 0 & 0 & a-b & a-b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Thus $\text{null}(A) = \text{span}\{(1, -1, 1, -1)^T\}$.
2. Let s_i denote the i-th column of A. Note that $s_4 = s_1 - s_2 + s_3$, and that $\{s_1, s_2, s_3\}$ is linearly independent for $a \neq b$. Thus $\{s_1, s_2, s_3\}$ is a basis of range(A).
3. Note that $(a-b)(1, -1, 1, -1)^T = (a-b, b-a, a-b, b-a)^T = s_1 + s_2 - s_3$. Thus $\{(1, -1, 1, -1)^T\}$ is a basis for S.

Problem 2. Let V be a finite-dimensional real inner product space. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V that is invariant under T.

1. Show that $U^\perp$ is invariant under T^*.
2. Construct an example of a $T \in \mathcal{L}(V)$ with a subspace U for which U is invariant under T but $U^\perp$ is not invariant under T. In your answer, give V, T and U, then show that U is invariant under T and show that $U^\perp$ is not invariant under T.

Solution:

1. Let $w \in U^\perp$, and let $u \in U$, so $Tu \in U$. Then $0 = \langle w, T(u)\rangle = \langle T^*(w), u\rangle$ for all $u \in U$ and $w \in U^\perp$, implying $U^\perp$ is T^*–invariant.
2. Define $T \in \mathbb{R}^2$ by $T(w, z) = (z, 0)$. First, $T(w, 0) = (0, 0)$ for all $w \in \mathbb{R}$. So $U = \{(w, 0) : w \in \mathbb{R}\}$ is T–invariant. With usual inner product, $U^\perp = \{(0, z) : z \in \mathbb{R}\}$. But $T(0, z) = (z, 0)$, so $U^\perp$ is not T-invariant.

Problem 3. Let A be a Hermitian matrix over $\mathbb{C}$ that is positive and invertible. (Such a matrix A is often called "Hermitian positive definite".) And let B be a Hermitian matrix over $\mathbb{C}$.

1. (10 points) Show that there exists an invertible matrix P such that $P^H AP = I$ and $P^H BP$ is diagonal.
   (Hint: First, show that there exists an invertible matrix T such that $A = T^H T$.)
2. (10 points) If B is positive, (such a matrix B is often called "Hermitian positive semidefinite",) show that
$$\det(A + B) \geq \det(A).$$

Solution:

1. Since A is positive definite, there exists an invertible matrix T such that $A = T^H T$. $T^{-H} BT^{-1}$ is Hermitian, so is diagonalizable. That is, there exists a unitary matrix U and a diagonal matrix D such that $U^H T^{-H} BT^{-1} U = D$. Let $P = T^{-1} U$. Then $P^H BP = D$, and
$$P^H AP = U^H T^{-H} (T^H T) T^{-1} U = U^H U = I.$$

2. Let P and D be as defined above. Then $A = P^{-H}P^{-1}$ and $B = P^{-H}DP^{-1}$. Since B is positive semidefinite, then the diagonal entries in D are nonnegative. Thus

$$\begin{aligned}\det(A+B) &= \det\left(P^{-H}(I+D)P^{-1}\right) = \det\left(P^{-H}P^{-1}\right)\det(I+D) \\ &= \det A \, \det(I+D) \geq \det A.\end{aligned}$$

Problem 4. Let V be a vector space over the field $\mathbb{F}$. For any $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$, let $G(\lambda, T)$ denote the generalized eigenspace of T corresponding to λ. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
(Hint: first show that $G(\lambda, T) \subseteq G(\frac{1}{\lambda}, T^{-1})$.)

Solution: Let $n = \dim V$. We first show $G(\lambda, T) \subseteq G(\frac{1}{\lambda}, T^{-1})$. Suppose $v \in G(\lambda, T)$. Then $(T - \lambda I)^n v = 0$. Note that the operators T^{-1} and $(T - \lambda I)$ commute, so

$$\begin{aligned}0 &= (T^{-1})^n (T-\lambda I)^n v \\ &= T^{-1}(T-\lambda I)T^{-1}(T-\lambda I)\cdots T^{-1}(T-\lambda I)v \\ &= (I - \lambda T^{-1})\cdots(I-\lambda T^{-1})v \\ &= (-\lambda)^n \left(T^{-1} - \frac{1}{\lambda}I\right)^n v.\end{aligned}$$

Thus, $(T^{-1} - \frac{1}{\lambda}I)^n v = 0$, so $v \in G(\frac{1}{\lambda}, T^{-1})$. Hence, $G(\lambda, T) \subset G\left(\frac{1}{\lambda}, T^{-1}\right)$. Replacing λ by $\frac{1}{\lambda}$, and T by T^{-1}, we have $G\left(\frac{1}{\lambda}, T^{-1}\right) \subseteq G(\lambda, T)$. Therefore, $G(\lambda, T) = G\left(\frac{1}{\lambda}, T^{-1}\right)$

Part II. Work **two** of problems 5 through 8.

Problem 5. Let $A \in \mathbb{R}^{m\times n}$ with $m \le n$.

1. (8 points) Prove that A is full rank if and only if AA^T is invertible.
2. (12 points) Let A now be of full rank. Prove that the matrix $P = I - A^T(AA^T)^{-1}A$ is the orthogonal projection matrix of $\mathbb{R}^n$ onto null (A).

Solution

1. ($\Rightarrow$) Suppose A has full (row) rank. Then A^T has full column rank. To see that AA^T is invertible, it suffices to show that if $AA^Tx = 0$, then $x = 0$. If $AA^Tx = 0$, then
$$0 = x^TAA^Tx = (A^Tx)^T(A^Tx) = \|A^Tx\|^2.$$
This implies $A^Tx = 0$. Since A^T has full column rank this implies $x = 0$.

   ($\Leftarrow$) Suppose AA^T is invertible. To see that A^T has full column rank m, note that if $A^Tx = 0$, then $AA^Tx = 0$ and thus $\text{null}(A^T) \subset \text{null}(AA^T)$, which implies that $\text{rank}(A^T) \ge \text{rank}(AA^T)$. As AA^T is invertible, it has full rank m. Thus $m \ge \text{rank}(A^T) \ge \text{rank}(AA^T) = m$. Thus A^T has full rank.

2. P has to satisfy two properties that we verify:

   - P projects onto null (A), i.e., $Px \in$ null (A) for any $x \in \mathbb{R}^n$:
   $$A \cdot Px = A \cdot (I - A^T(AA^T)^{-1}A)x = Ax - Ax = 0$$
   - P projects orthogonally to null (A), i.e., $d^T(x - Px) = 0$ for all $d \in$ null (A) (note $Ad = 0 = d^TA^T$):
   $$d^T(x - Px) = d^Tx - d^T(I - A^T(AA^T)^{-1}A)x = d^TA^T \cdot (AA^T)^{-1}Ax = 0$$

Problem 6. Let V be a finite-dimensional inner product space. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $v_1, \dots, v_n$ are vectors in V such that
$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$
for each j. Prove that $v_1, \dots, v_n$ is a basis of V.

Hint #1: First prove that for any n scalars a_j, $j = 1, \ldots, n$, we have

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j| \le \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}.$$

Hint #2: Assume a_1, a_2, ..., a_n are scalars such that $\sum_{j=1}^{n} a_j v_j = 0$ and look at $\left\| \sum_{j=1}^{n} a_j (e_j - v_j) \right\|$.

Solution We can prove the first hint using Cauchy-Schwartz on

$$u = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} |a_1| \\ \vdots \\ |a_n| \end{pmatrix}$$

with the standard Euclidean inner product. Cauchy-Schwartz says that $|u^T v| \le ||u||_2 ||v||_2$. We have that

$$|u^T v| = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j|, \quad \|u\|_2 = 1, \quad \text{and} \quad \|v\|_2 = \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}.$$

So that proves

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j| \le \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}. \tag{1}$$

To prove that $v_1, \ldots, v_n$ is a basis, it is sufficient to prove that $v_1, \ldots, v_n$ is linearly independent.

For sake of contradiction, suppose $v_1, \ldots, v_n$ is linearly dependent. Then there exist scalars a_1, a_2, ..., a_n not all zeros such that

$$\sum_{j=1}^{n} a_j v_j = 0.$$

Following hint #2, observe that

$$\left\| \sum_{j=1}^{n} a_j (e_j - v_j) \right\| = \left\| \sum_{j=1}^{n} a_j e_j - \sum_{j=1}^{n} a_j v_j \right\| = \left\| \sum_{j=1}^{n} a_j e_j \right\|,$$

where the second equality comes from the fact that $\sum_{j=1}^{n} a_j v_j = 0$.

Since e_1, ..., e_n is orthonormal, we get $\left\|\sum_{j=1}^{n} a_j e_j\right\| = \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}$, so

$$\left\|\sum_{j=1}^{n} a_j(e_j - v_j)\right\| = \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}. \tag{2}$$

On the other hand, using the triangle inequality, we see that

$$\left\|\sum_{j=1}^{n} a_j(e_j - v_j)\right\| \leq \sum_{j=1}^{n} |a_j|\|e_j - v_j\|. \tag{3}$$

And using the assumption

$$\forall j, \quad \|e_j - v_j\| < \frac{1}{\sqrt{n}},$$

we get that

$$\forall j, \quad |a_j|\|e_j - v_j\| \leq \frac{1}{\sqrt{n}}|a_j|,$$

Note that the $<$ is transformed to a $\leq$ because $|a_j|$ may be zero. However because at least one of the a_j's is not 0, we have that

$$\exists j, \quad |a_j|\|e_j - v_j\| < \frac{1}{\sqrt{n}}|a_j|,$$

And so, we get that

$$\sum_{j=1}^{n} |a_j|\|e_j - v_j\| < \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j|.$$

Unsing this last relation into Equation (3), we get

$$\left\|\sum_{j=1}^{n} a_j(e_j - v_j)\right\| < \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j| \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}, \tag{4}$$

where the second inequality comes from Equation (1).

We now see that we have a contradiction between Equations (4) and (2). Thus, our assumption "$v_1, \ldots, v_n$ is linearly dependent" is false. Therefore v_1, ..., v_n is linearly independent, so is a basis of V.

Problem 7. Let $A, B \in \mathbb{R}^{n\times n}$. Two matrices A, B are called simultaneously diagonalizable if there exists an invertible matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

1. (6 points) Prove that if A, B are simultaneously diagonalizable then $AB = BA$.
2. (14 points) Prove that if $AB = BA$ and if one of the matrices has n distinct eigenvalues then A, B are simultaneously diagonalizable.

Solution

1. Let $D_A = S^{-1}AS$ and $D_B = S^{-1}BS$ be the two corresponding diagonal matrices. Clearly, $D_A D_B = D_B D_A$, and thus $AB = SD_AS^{-1}SD_BS^{-1} = SD_AD_BS^{-1} = SD_BD_AS^{-1} = SD_BS^{-1}SD_AS^{-1} = BA$.
2. Without loss of generality, let A have n distinct eigenvalues. Then A is diagonalizable, i.e., there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $D_A = S^{-1}AS = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. We define $D_B := S^{-1}BS$. As $AB = BA$, one obtains $D_AD_B = S^{-1}ASS^{-1}BS = S^{-1}ABS = S^{-1}BAS = S^{-1}BSS^{-1}AS = D_BD_A$. Let $D_AD_B = (c_{ij})_{ij}$ and $D_BD_A = (d_{ij})_{ij}$.

   Then it holds that $c_{ii} = d_{ii}$ is the λ_i-multiple of the corresponding entry of D_B, but for $i \neq j$ it holds that c_{ij} is the λ_i-multiple and that d_{ij} is the λ_j-multiple of the corresponding entry of D_B, respectively. As the eigenvalues λ_i, λ_j of A are distinct and as $D_AD_B = D_BD_A$, D_B has to be a diagonal matrix.

Problem 8.

Let A and B in $\mathbb{R}^{n\times n}$ such that $AB - BA = A$.

1. (10 points) Prove that $A^kB - BA^k = kA^k$
2. (10 points) Prove that A is nilpotent.

Solution The first part is done by induction on k for the statement $A^kB - BA^k = kA^k$. For $k = 1$, this is true by initial assumption. Assume the statement is true for k, let us prove it is true for $k + 1$. We have that $A^kB - BA^k = kA^k$. Multiplying by A on the left and we get (1): $A^{k+1}B - ABA^k = kA^{k+1}$. We also have that $AB - BA = A$, so multiplying by A^k on the right and we get (2): $ABA^k - BA^{k+1} = A^{k+1}$. Now adding (1) and (2) together leads to the desired result that is $A^{k+1}B - BA^{k+1} = (k + 1)A^{k+1}$.

The second part is done by arguing that part 1 shows that A^k is an eigenvector of eigenvalue k of $\Phi(A) = A^kB - BA^k$. If A^k is not zero, then Φ has an infinity of eigenvalues (1, 2, etc.) which is absurd since the dimension of our space is n^2.
