University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Aug. 11, 2023

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Stephen Billups, Yaning Liu (Chair), Dmitriy Ostrovskiy

Part I. Work **all** of problems 1 through 4.

Problem 1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$. Prove that $\dim U - \dim V \leq \dim \operatorname{null} ST - \dim \operatorname{null} S$.

Problem 2.

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be 3 unit vectors in a real inner-product space V.

(a) (15 points) Show that $2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle \geq \langle \mathbf{u}, \mathbf{v}\rangle^2 + \langle \mathbf{u}, \mathbf{w}\rangle^2 + \langle \mathbf{v}, \mathbf{w}\rangle^2 - 1$. Hint: apply the first step of the Gram-Schmidt process to vectors $\mathbf{v}$ and $\mathbf{w}$ with respect to $\mathbf{u}$ and and apply the Cauchy-Schwarz inequality to the resulting pair of vectors.

(b) (5 points) Show that the equality is reached if and only if vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are linearly dependent.

Problem 3. Let T be a positive operator on V. Suppose $v, w \in V$ are such that $Tv = w$ and $Tw = v$. Prove that $v = w$.

Problem 4. Let $n \geq 2$.

(a) Is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ and $A^n = 0$? Give an example to show such a matrix exists (and explain why the matrix satisfies the two conditions), or disprove it.

(b) Show that an $n \times n$ upper triangular matrix A with $A^n \neq 0$ and $A^{n+1} = 0$ does not exist.

Part II. Work **two** of problems 5 through 8.

Problem 5. Let T be a linear map on a vector space V, $\dim V = n$.

(a) If for some vector $\boldsymbol{v}$, the vectors $\boldsymbol{v}$, $T(\boldsymbol{v})$, $T^2(\boldsymbol{v})$, $\ldots$, $T^{n-1}(\boldsymbol{v})$ are linearly independent, show that every eigenvalue of T has only one corresponding eigenvector up to a scalar multiplication.

(b) If T has n distinct eigenvalues, and vector $\boldsymbol{u}$ is the sum of n eigenvectors, corresponding to the distinct eigenvalues, show that $\boldsymbol{u}$, $T(\boldsymbol{u})$, $T^2(\boldsymbol{u})$, $\ldots$, $T^{n-1}(\boldsymbol{u})$ are linearly independent (and thus form a basis of V).

Problem 6. Let A be an $n \times n$ positive semidefinite matrix.

(a) Show that

$$\left\|(I - A)(I + A)^{-1}\boldsymbol{x}\right\|_2 \leq \|\boldsymbol{x}\|_2 \,, \; \boldsymbol{x} \in \mathbb{C}^n.$$

(b) Show that $\boldsymbol{x} \in \operatorname{null} A$ is equivalent to

$$(I - A)(I + A)^{-1}\boldsymbol{x} = \boldsymbol{x}.$$

Problem 7. Let A be an isometry on a finite-dimensional real inner product space V which satisfies $A^2 = -I$. Prove that for every vector $\mathbf{v}$ in V, $A\mathbf{v}$ is orthogonal to $\mathbf{v}$.

Problem 8. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a set of vectors in a real inner-product space such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle < 0$ for all $i \neq j$.

(a) (5 points) Show that any linear combination of a set of vectors can be written as a difference of two linear combinations with non-negative coefficients.

(b) (7 points) If set S is linearly dependent, show that any nontrivial linear combination of vectors from S equal to $\mathbf{0}$ contains only coefficients of the same sign (disregarding zeros).

(c) (8 points) Show that $\dim \operatorname{span} S \geq n - 1$.
