University of Colorado Denver
Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
June 12, 2023

Student number (not your name): ______________________________
Exam Rules:

- This is a closed book exam. You may use one page of notes (1 side of a letter-sized piece of paper). You may not use any other external aides during the exam, such as
  - communicating with anyone other than the exam proctor;
  - consulting the internet, textbooks, solutions of previous exams, etc.
  - using calculators or mathematical software.
- You have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Each problem is worth 20 points. The weights for each part on multi-step problems are indicated in the problem.
- Be sure to show all work that is relevant for each problem, but do not turn in scratch work.
- Justify your solutions: **cite theorems that you use**, justify that their assumptions are satisfied, provide specific counter-examples for disproof, give explanations, and show calculations for numerical computations. If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask the proctor. This exam uses the definitions from Pugh. If you want to use definitions from Rudin, please state them and use them consistently.
- When read aloud, your solution must make complete English sentences. Do not put in just math symbols and expect the committee to guess the rest.
- Begin each solution on a new page. Put your student number (not your name) and page number on the top of every page.
- Write legibly using a dark pencil or pen. Write on only one side of the paper, the back side will not be scanned. Leave 1in margins, the scanner won't pick up your writing all the way to the edges. If you leave a reasonable space between the lines for our notes, the committee will be much happier!
- In case of a major disruption due to which the exam cannot be completed, for example due to health reasons or a campus evacuation, students are entitled to a choice between acceptance of partial work and a partial new problem set, or a full new problem set.

Part 1: Solve all problems 1-4.

1. Suppose (X, d) is a metric space. Denote by $\overline{S}$ the closure of a set $S \subset X$. Suppose $A_1, A_2, \ldots \subset X$.

   (a) (10 points) Prove that $\bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n}$.

   (b) (10 points) Give an example where the inclusion is proper.

   Solution:

   (a) Suppose $x \in \bigcup_{n=1}^{\infty} \overline{A_n}$. Then, there exists n such that $x \in \overline{A_n}$. Then $x = \lim_{k\to\infty} x_k$ for some sequence $(x_k) \subset A_k$. Since $(x_k) \subset A_k \subset \bigcup_{n=1}^{\infty} A_n$, it follows that $x \in \overline{\bigcup_{n=1}^{\infty} A_n}$.
   Another solution: Use the definition of $\overline{S}$ as the smallest closed set containing S, that is, $\overline{S} = \bigcap_{F\supset S,\ F\subset X \text{ closed}} F$. For any $k = 1, 2, \ldots$, it holds that $A_k \subset \bigcup_{n=1}^{\infty} A_n \subset \overline{\bigcup_{n=1}^{\infty} A_n}$. Since the intersection of a family of closed sets is closed, $\overline{\bigcup_{n=1}^{\infty} A_n}$ is closed. Thus, $\overline{\bigcup_{n=1}^{\infty} A_n}$ is one of the closed sets in the definition of closure of A_k. It follows that $\overline{A_k} \subset \overline{\bigcup_{n=1}^{\infty} A_n}$ for all k, and thus $\bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n}$.

   (b) Consider $X = \mathbb{R}$ with the Euclidean metric, and $A_n = \{1/n\}$, $n = 1, 2, \ldots$. The sets A_n are singletons, therefore closed, therefore $A_n = \overline{A_n}$, so $\bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} A_n = \{1/n : n = 1, 2, \ldots\}$. But the point 0 is a cluster point of the set $\{1/n : n = 1, 2, \ldots\}$ because $\lim_{n\to\infty} 1/n = 0$, so $0 \in \overline{\bigcup_{n=1}^{\infty} A_n}$, but $0 \notin A_n$ for any n, so $0 \notin \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} A_n$. We can conclude that $\bigcup_{n=1}^{\infty} \overline{A_n} \subsetneq \overline{\bigcup_{n=1}^{\infty} A_n}$.

2. Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous functions between metric spaces X, Y, Z. Prove that $h : X \to Z$ with $h(x) = g(f(x))$ is uniformly continuous.

   Solution:

   Let $\varepsilon > 0$. Choose $\rho > 0$ such that $d_Z(g(y_1), g(y_2)) < \varepsilon$ whenever $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) < \rho$ (possible since g is uniformly continuous). Further, choose $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \rho$ whenever $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ (possible since f is uniformly continuous.).

   Now let $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$. Then $d_Y(f(x_1), f(x_2)) < \rho$, and thus $d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon$.

3. Suppose f is a bounded real function on $[a, b]$ such that f^2 is Riemann integrable. Does it follow that f is Riemann integrable? Does the answer change if we assume that f^3 is Riemann integrable

   Solution:

   All statements below are understood on $[a, b]$.

   f does not have to be Riemann integrable. For example, define $f(x) = 1$ if x is rational, and $f(x) = -1$ if x is irrational. Then f^2 is a constant function, so it is Riemann integrable.

On the other hand, if f^3 is Riemann integrable, then so is f. To see this, define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = \sqrt[3]{x}$. Then $f = \phi \circ f^3$. Since ϕ is continuous and f^3 is Riemann integrable, it follows that the composition $f = \phi \circ f^3$ is Riemann integrable.

4. Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and the sequence (f_n) is equicontinuous on $[0, 1]$. Show that f is constant on $[0, \infty)$.

Solution:

Let x, y be distinct points in $[0, \infty)$ and let $\epsilon > 0$ be given. By equicontinuity, there exists $\delta > 0$ such that for any $a, b \in [0, 1]$, if $|a - b| < \delta$, then $|f_n(a) - f_n(b)| < \epsilon$. Choose n large enough so that $|x/n - y/n| < \delta$ and $x/n,\ y/n \in [0, 1]$. Then,

$$|f(x) - f(y)| = \left| f_n\left(\frac{x}{n}\right) - f_n\left(\frac{y}{n}\right) \right| < \epsilon.$$

Since this holds for every $\epsilon > 0$, it follows that $f(x) = f(y)$. Since this is true for every $x, y \in [0, \infty)$, we conclude that f is constant on $[0, \infty)$.

Part 2 - Solve 2 out of the following 4 problems.

5. Prove this version of Lebesgue's Number Lemma (without using the lemma itself):

Let (X, d) be a compact metric space with an open cover $\mathcal{U}$. Then there exists a $\delta > 0$ such that every open δ-ball

$$B(x, \delta) = \{y \in X : d(x, y) < \delta\}$$

is contained in some element of $\mathcal{U}$.

Solution:

Definition (Pugh, p. 79): A subset A of a metric space M is (sequentially) compact if every sequence $(a_n) \subset A$ has a subsequence (a_{n_k}) that converges to a limit in A.

Suppose for the sake of contradiction, that no such δ exists. Therefore, for every element of the sequence $\delta_n = \frac{1}{n}$, we can find an $x_n \in X$ such that $B(x_n, \frac{1}{n}) \not\subset U$ for every $U \in \mathcal{U}$, yielding a sequence $(x_n) \subset X$. Since X is compact, there exists a convergent subsequence $x_{n_k} \to x$ for some $x \in X$. Since $\mathcal{U}$ is a cover, there exists an $U \in \mathcal{U}$ with $x \in U$. Since U is open, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. But for a large enough k, we have that $d(x, x_{n_k}) < \frac{1}{2}\varepsilon$, and $\frac{1}{n_k} < \frac{1}{2}\varepsilon$, which implies that $B(x_{n_k}, \frac{1}{n_k}) \subset B(x, \varepsilon) \subset U$ by the triangle inequality: Let $y \in B(x_{n_k}, \frac{1}{n_k})$. Then,

$$d(x, y) \leq d(x, x_{n_k}) + d(x_{n_k}, y) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

But $B(x_{n_k}, \frac{1}{n_k}) \subset U$ is a contradiction.

6. Suppose that $A, B \subset \mathbb{R}$ and define $A + B = \{z = x + y : x \in A, y \in B\}$.

   (a) (5 points) Prove that if a is an upper bound on A and b is an upper bound on B, then $a + b$ is an upper bound on $A + B$.

(b) (3 points) Prove that if $X \subset \mathbb{R}$, $X \neq \emptyset$, then $\sup X > -\infty$.

(c) (2 points) Prove that if $A \neq \emptyset$ and $B \neq \emptyset$, then $\sup A + \sup B$ is defined

(d) (10 points) Prove that if $A \neq \emptyset$ and $B \neq \emptyset$, then $\sup(A + B) = \sup A + \sup B$.

Solution:

(a) Suppose a is an upper bound on A and b is an upper bound on B, that is,

$$\forall x \in A : x \leq a \text{ and } \forall y \in A : y \leq b.$$

Let $z \in A + B$. Then $z = x + y : x \in A, y \in B$, and since $x \leq a$ and $y \leq b$, it follows that

$$z = x + y \leq a + b.$$

Thus, $a + b$ is an upper bound on $A + B$.

(b) Since $X \neq \emptyset$, there exists $x \in X$. Since $X \subset \mathbb{R}$ and $-\infty \notin \mathbb{R}$, it holds that $x \neq -\infty$. If X has an upper bound, then, by definition, sup X is the least upper bound on X, and from the completeness of reals, sup X exists. Since sup X is an upper bound on X and $x \in X$, we have sup $X \geq x > -\infty$. If X does not have an upper bound, then $\sup = +\infty > -\infty$.

(c) Set $a = \sup A$ and $b = \sup B$. Since $a, b > -\infty$ by part b, $a + b$ is defined (the case $\infty - \infty$ cannot happen).

(d) From the completeness of reals, $\sup(A + B)$ is known to exist. Set $a = \sup A$ and $b = \sup B$. If either $a = \infty$ or $b = \infty$, then $a + b = \infty \geq \sup(A + B)$. Otherwise, a is an upper bound on A and b is an upper bound on B, and we have from part a, that $a + b$ is an upper bound on $A + B$. From the definition of sup, $\sup(A + B)$ is the least upper bound on $A + B$, thus $\sup(A + B) \leq a + b$.

To show the opposite inequality, first consider the case when A is not bounded above. Since $B \neq \emptyset$, there exists $\bar{y} \in B$ and by the definition of $A + B$, $A + B \supset C$, where $C = \{z = x + \bar{y} : x \in A\}$ is not bounded above, because A is not bounded above. Thus, $A + B$ is not bounded above, and $\sup(A + B) = \infty \geq \sup A + \sup B$. The case when B is not bounded above follows by swapping the notation of A and B.

It remains to consider the case when both A and B are bounded above. Then, $\sup A \in \mathbb{R}$ and $\sup B \in \mathbb{R}$. Let $\varepsilon > 0$. Since $\sup A \in \mathbb{R}$, it holds that $\sup A - \varepsilon/2 < \sup A$, and from the definition of supremum, there exist $x \in A$ such that $x > \sup A - \varepsilon/2$. Swapping the notation A and B, we have that there exist $y \in B$ such that $y > \sup B - \varepsilon/2$. Then $x + y \in A + B$ and $x + y > \sup A + \sup B - \varepsilon$, thus $\sup(A + B) > \sup A + \sup B - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\sup(A + B) \geq \sup A + \sup B$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\lim_{x \to -\infty} f(x) = \alpha$ and $\lim_{x \to \infty} = \beta$, where α and β are finite. Prove that f is uniformly continuous.

Solution:

Let $\epsilon > 0$ be given. Since $\lim_{x \to -\infty} = \alpha$, there exists $L < 0$ such that $|f(x) - \alpha| < \epsilon/2$ for all $x \leq L$. Similarly, there exists $U \geq 0$ such that $|f(x) - \beta| < \epsilon/2$ for all $x \geq U$. Without loss of generality, assume $L < U$. Since f is continuous, it is uniformly continuous on the compact set $[L-1, U+1]$. Thus, there exists $\delta > 0$ such that if $x, y \in [L-1, U+1]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon/2$. Without loss of generality, assume $\delta < 1$.

Suppose $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta$. Consider the following 3 cases:

Case 1: $x, y \in [L-1, U+1]$:

By the definition of δ, $|f(x) - f(y)| < \epsilon/2 < \epsilon$.

Case 2: $x, y \leq L$:

If $x, y \leq L$, then $|f(x) - f(y)| \leq |f(x) - \alpha| + |\alpha - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$.

Case 3: $x, y \geq U$:

If $x, y \geq U$, then $|f(x) - f(y)| \leq |f(x) - \beta| + |\beta - f(y)| < \epsilon/2 + \epsilon/2 < \epsilon$.

Since $|x - y| < \delta < 1$, at least one of the 3 cases above must hold. Thus, $|f(x) - f(y)| < \epsilon$, which shows that f is uniformly continuous.

8. Let (f_n) be a uniformly bounded sequence of functions that are Riemann integrable on $[a, b]$, and define

$$F_n(x) = \int_a^x f_n(t)dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ that converges uniformly on $[a, b]$.

Solution:

By assumption, there exists $M > 0$ such that $|f_n(x)| \leq M$ for all n and x. Thus, $|F_n(x)| \leq \int_a^x M dt \leq M(b-a)$ for all n and $x \in [a, b]$. Hence (F_n) is uniformly bounded on $[a, b]$.

Given $\epsilon > 0$, let $\delta = \epsilon/M$. Then, for any $x, y \in [a, b]$, if $|x - y| < \delta$, then

$$|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t)dt \right| \leq M|x - y| < \epsilon.$$

Hence (F_n) is equicontinuous. Therefore, by Arzelà-Ascoli theorem, (F_n) has a subsequence that converges uniformly on $[a, b]$.