University of Colorado Denver
Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
June 12, 2023

Student number (not your name): ______________________________

Exam Rules:

- This is a closed book exam. You may use one page of notes (1 side of a letter-sized piece of paper). You may not use any other external aides during the exam, such as
  - communicating with anyone other than the exam proctor;
  - consulting the internet, textbooks, solutions of previous exams, etc.
  - using calculators or mathematical software.
- You have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Each problem is worth 20 points. The weights for each part on multi-step problems are indicated in the problem.
- Be sure to show all work that is relevant for each problem, but do not turn in scratch work.
- Justify your solutions: **cite theorems that you use**, justify that their assumptions are satisfied, provide specific counter-examples for disproof, give explanations, and show calculations for numerical computations. If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask the proctor. This exam uses the definitions from Pugh. If you want to use definitions from Rudin, please state them and use them consistently.
- When read aloud, your solution must make complete English sentences. Do not put in just math symbols and expect the committee to guess the rest.
- Begin each solution on a new page. Put your student number (not your name) and page number on the top of every page.
- Write legibly using a dark pencil or pen. Write on only one side of the paper, the back side will not be scanned. Leave 1in margins, the scanner won't pick up your writing all the way to the edges. If you leave a reasonable space between the lines for our notes, the committee will be much happier!
- In case of a major disruption due to which the exam cannot be completed, for example due to health reasons or a campus evacuation, students are entitled to a choice between acceptance of partial work and a partial new problem set, or a full new problem set.

Part 1: Solve all problems 1-4.

1. Suppose (X, d) is a metric space. Denote by $\overline{S}$ the closure of a set $S \subset X$. Suppose $A_1, A_2, \ldots \subset X$.

   (a) (10 points) Prove that $\bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n}$.

   (b) (10 points) Give an example where the inclusion is proper.

2. Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous functions between metric spaces X, Y, Z. Prove that $h : X \to Z$ with $h(x) = g(f(x))$ is uniformly continuous.

3. Suppose f is a bounded real function on $[a, b]$ such that f^2 is Riemann integrable. Does it follow that f is Riemann integrable? Does the answer change if we assume that f^3 is Riemann integrable

4. Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and the sequence (f_n) is equicontinuous on $[0, 1]$. Show that f is constant on $[0, \infty)$.

Part 2 - Solve 2 out of the following 4 problems.

5. Prove this version of Lebesgue's Number Lemma (without using the lemma itself):

   Let (X, d) be a compact metric space with an open cover $\mathcal{U}$. Then there exists a $\delta > 0$ such that every open δ-ball

   $$B(x, \delta) = \{y \in X : d(x, y) < \delta\}$$

   is contained in some element of $\mathcal{U}$.

6. Suppose that $A, B \subset \mathbb{R}$ and define $A + B = \{z = x + y : x \in A, y \in B\}$.

   (a) (5 points) Prove that if a is an upper bound on A and b is an upper bound on B, then $a + b$ is an upper bound on $A + B$.

   (b) (3 points) Prove that if $X \subset \mathbb{R}$, $X \neq \emptyset$, then $\sup X > -\infty$.

   (c) (2 points) Prove that if $A \neq \emptyset$ and $B \neq \emptyset$, then $\sup A + \sup B$ is defined

   (d) (10 points) Prove that if $A \neq \emptyset$ and $B \neq \emptyset$, then $\sup (A + B) = \sup A + \sup B$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\lim_{x \to -\infty} f(x) = \alpha$ and $\lim_{x \to \infty} = \beta$, where α and β are finite. Prove that f is uniformly continuous.

8. Let (f_n) be a uniformly bounded sequence of functions that are Riemann integrable on $[a, b]$, and define

   $$F_n(x) = \int_a^x f_n(t)dt \quad (a \leq x \leq b).$$

   Prove that there exists a subsequence $\{F_{n_k}\}$ that converges uniformly on $[a, b]$.