University of Colorado Denver
Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
January 20, 2023

Student number (not your name): ______________________________

Exam Rules:

- This is a closed book exam. You may not use external aides during the exam, such as
  - communicating with anyone other than the exam proctor;
  - consulting the internet, textbooks, solutions of previous exams, etc.
  - using calculators or mathematical software.
- You have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Each problem is worth 20 points. The weights for each part on multi-step problems are indicated in the problem.
- Be sure to show all work that is relevant for each problem, but do not turn in scratch work.
- Justify your solutions: **cite theorems that you use**, justify that their assumptions are satisfied, provide specific counter-examples for disproof, give explanations, and show calculations for numerical computations.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask the proctor.
- This exam uses the definitions from Pugh. If you want to use definitions from Rudin, please state them and use them consistently.
- Begin each solution on a new page and write on only one side of the paper. Put your student number (not your name) and page number on the top of every page. Write legibly using a dark pencil or pen.
- In case of a major disruption due to which the exam cannot be completed, for example due to health reasons or a campus evacuation, students are entitled to a choice between acceptance of partial work and a partial new problem set, or a full new problem set.

Part 1: Solve all problems 1-4.

1. Construct a compact subset of $\mathbb{R}$ with a denumerable set of cluster points. (Definitions: y is a cluster point of A if every neighborhood of y contains an element of A besides y, or, equivalently, infinitely many points of A. Denumerable set is countable and infinite.)

   Solution. Example: Define

$$A = \left\{ a_{in} : a_{in} = \frac{1}{n} + \frac{1}{i}\left(\frac{1}{n} - \frac{1}{n+1}\right), \quad i, n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

   Then $\lim_{i\to\infty} a_{in} = \frac{1}{n}$ for all n, so $\frac{1}{n}$ are cluster points of S. The point 0 is also a cluster point, since $\lim_{n\to\infty} \frac{1}{n} = 0$.

   We need to show that no other points are cluster points of S. We have

$$\frac{1}{n} < a_{n1} < 2 \qquad \text{for } n = 1 \text{ and all } i \in \mathbb{N}$$

   and

$$\frac{1}{n} < a_{in} < \frac{1}{n-1} \qquad \text{for all } n > 1 \text{ and all } i \in \mathbb{N},$$

   because $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n-1} - \frac{1}{n}$. Thus, the sets $A_n = \{a_{in} : i \in \mathbb{N}\}$ are contained in disjoint intervals. Since for each n, the set A_n has no cluster points other than $\frac{1}{n}$, the cluster points of S are exactly $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, a countable set. The set A is closed since it contains its cluster points, it is bounded since $A \subset [0, 2]$, therefore it is compact.

2. Let (X, d) be a metric space and f and g be continuous maps $f, g : X \to \mathbb{R}$. Let E be a dense subset of X.

   (a) (10 points) Prove that $f(E)$ is dense in $f(X)$.

   (b) (10 points) If $g(x) = f(x)$ for all $x \in E$, prove that $g(x) = f(x)$ for all $x \in X$.

Solution.

(a) Definitions from Pugh: If $S \subset M$ and $\overline{S} = M$ then S is dense in M. Closure $\overline{S} = \{p \in M : p \text{ is a limit of } S\}$. A point p is a limit of S if there exists a sequence (p_n) in S that converges to it.

In summary, S is dense in M if for every $p \in M$ exists a sequence $(p_n) \subset S$ such that $p_n \to p$.

Let $y \in f(X)$. Then there exists $x \in X$ such that $f(x) = y$. Since E is dense in X, there exists sequence $(x_n) \subset E$ such that $x_n \to x$. Define $y_n = f(x_n)$. Then $y_n \in f(E)$ and $y_n \to y$ because f is continuous.

(b) Let $x \in X$. Since E is dense in X, there exists sequence $(x_n) \subset E$ such that $x_n \to x$. Since f is continuous, $f(x_n) \to f(x)$ and since g is continuous, $g(x_n) \to g(x)$. Since $x_n \in E$, $f(x_n) = g(x_n)$, so from the uniqueness of limit, $f(x) = g(x)$.

3. Let (a_n) and (b_n) be bounded nonnegative sequences. Prove that

$$\liminf_{n\to\infty}(a_n b_n) \geq (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n).$$

Solution: Let $m \in \mathbb{N}$. Denote $A = \inf\{a_n : n \geq m\}$ and $B = \inf\{b_n : n \geq m\}$. Since $A \geq 0$ and $B \geq 0$, it follows that $a_n b_n \geq AB$ for all $n \geq m$,. Therefore,

$$\inf\{a_n b_n : n \geq m\} \geq \inf\{a_n : n \geq m\} \inf\{b_n : n \geq m\}.$$

Taking the limit for $m \to \infty$, we get

$$\begin{aligned}
\liminf_{n\to\infty}(a_n b_n) &= \lim_{m\to\infty} \inf\{a_n b_n : n \geq m\} \\
&\geq \lim_{m\to\infty} \inf\{a_n : n \geq m\} \inf\{b_n : n \geq m\} \\
&= \lim_{m\to\infty} \inf\{a_n : n \geq m\} \lim_{m\to\infty} \inf\{b_n : n \geq m\} \\
&= (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n).
\end{aligned}$$

Another solution. Denote

$$a = \liminf_{n\to\infty} a_n, \quad b = \liminf_{n\to\infty} b_n.$$

The sequences (a_n) and (b_n) are assumed to be nonnegative and bounded, i.e., for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$0 \leq a_n \leq M, \quad 0 \leq b_n \leq M.$$

Let $\varepsilon > 0$. From the definition of lim inf and using the fact that a, b are finite because the sequences (a_n) and (b_n) are bounded, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a - \varepsilon < a_n, \quad b - \varepsilon < b_n,$$

and, consequently, for all $n \geq N$,

$$ab < (a_n + \varepsilon)(b_n + \varepsilon) \leq a_n b_n + 2M\varepsilon + \varepsilon^2.$$

Therefore,

$$ab \leq \liminf_{n\to\infty} a_n b_n + 2M\varepsilon + \varepsilon^2.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $ab \leq \liminf_{n\to\infty} a_n b_n$.

4. Let $f_n(x) = \sin(n + x)$, $x \in [0, 2\pi]$, $n = 1, 2, \ldots$. Prove that (f_n) has a pointwise convergent subsequence.

   Solution: The interval $[0, 2\pi]$ is closed and bounded. $f_n(x)| \leq 1$ for all $x \in [0, 2\pi]$, so (f_n) is uniformly bounded. Also, $|f_n'(x)| = |\cos(n + x)| \leq 1$, so (f_n) is equicontinuous. By Arzèla-Ascoli theorem, uniformly bounded and equicontinuous sequence of functions on a closed and bounded interval has a uniformly convergent subsequence. Since uniformly convergent sequence of functions is pointwise convergent, (f_n) has a pointwise convergent subsequence.

Part 2 - Solve 2 out of the following 4 problems.

5. Let $f : X \to \mathbb{R}$. Define the graph of f to be the set $G = \{(x, y) \in X \times \mathbb{R} : y = f(x)\}$. Prove:

   (a) (10 points) If f is continuous then G is closed.

   (b) (10 points) If f is continuous and X is compact, then G is compact.

Solution.

(a) $G \subset X \times \mathbb{R}$, so closed here means closed in the product metric space $X \times \mathbb{R}$ with the distance function $d_{X\times\mathbb{R}}((x, y), (x', y')) = d(x, x') + |y - y'|$. Let $(x_n, y_n) \subset G$ with $(x_n, y_n) \to (x, y)$. Thus, $x_n \to x$ and $y_n \to y$. Since f is continuous, $y_n = f(x_n) \to f(x)$. But $y_n \to y$, by uniqueness of the limit it follows that $f(x) = y$, which implies $(x, y) \in G$.

Note: Solutions assuming $X \subset \mathbb{R}$ and $G \subset \mathbb{R} \times \mathbb{R}$ were also accepted.

(b) Since f is continuous and X is compact, it follows that f is bounded. Thus, $f(X) \subset [a, b]$ for some $-\infty < a < b < \infty$, so $G \subset X \times [a, b]$, which is compact as the product of compact metric spaces. Since G is closed by part (a), it is a closed subset of a compact set, therefore compact.

6. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Darboux integrable functions which converge pointwise to a function $f : [0,1] \to \mathbb{R}$. Prove or find a counterexample:

The function f is Darboux integrable on $[0,1]$.

(Darboux integral as defined in Pugh is called Riemann integral in Rudin.)

Solution: Counterexample. Enumerate all rational numbers in $[0,1]$ by $q_1, q_2, q_3, \ldots$. Choose

$$f_n(x) = \begin{cases} 1, & \text{if } x = q_k \text{ for some } k < n, \\ 0, & \text{otherwise} \end{cases}$$

For each n, f_n has finitely many discontinuities and it is bounded, thus it is Darboux integrable. But $\lim_{n\to\infty} f_n = f$,

$$f(x) = \begin{cases} 1, & \text{if } x \text{ rational} \\ 0, & \text{if } x \text{ irrational} \end{cases}$$

which is known not to be Darboux integrable.

Another solution: Define

$$f_n(x) = \begin{cases} n^2 x \text{ if } x \in [0, 1/n] \\ 1/x \text{ if } x \in (1/n, 1] \end{cases}, \qquad f_n(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1/x \text{ if } x \in (0, 1] \end{cases}$$

Then $f_n(x) \to f(x)$ for all $x \in [0,1]$, f_n is continuous on $[0,1]$ thus Darboux integrable, but f is not bounded and thus not Darboux integrable on $[0,1]$.

7. Define an open mapping $f : X \to Y$ to be one where $f(V)$ is open in Y whenever V is open in X. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is open and continuous then it is monotone.

   Solution: Suppose f is not monotone. Then there is $a < b < c$ with $f(a) < f(b)$ and $f(b) > f(c)$, or the reverse. Suppose $f(a) < f(b)$ and $f(b) > f(c)$ (the other case is analogous). Since f is continuous, it attains its maximum on $[a, c]$ (Extreme Value Theorem), so there is $d \in (a, c)$ with $f(d) \geq f(x)$ for every $x \in (a, c)$. Thus, $f(d) \in f((a, c))$, but for every $\epsilon > 0$, $(f(d) - \epsilon, f(d) + \epsilon) \not\subset (f((a, c))$. Thus, $f((a, c))$ is not open, so f is not an open map.

8. Suppose that (X, d) is a metric space and (f_n) is a sequence of continuous functions $f_n : X \to \mathbb{R}$ convergent pointwise on X to a function f.

(a) (10 points) Prove that if $f_n \rightrightarrows f$ on X, then for every convergent sequence $(x_n) \subset X$, $\lim_{n\to\infty} f_n(x_n) = f(x)$, where $x = \lim_{n\to\infty} x_n$.

(b) (10 points) Is the converse true? Prove or provide a counterexample.

Solution.

(a) Let $x_n \to x$ in (X, d). Since $f_n \rightrightarrows f$ on X and f_n are continuous, f is continuous. Let $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|f(t) - f(x)| < \frac{\varepsilon}{2}$ for all $t \in X$ such that $d(x, t) < \delta$. Since $x_n \to x$, here exists $N_1 > 0$ such that for all $n \geq N_1$, $d(x_n, x) < \delta$. Since $f_n \rightrightarrows f$ on X, there exists $N_2 > 0$ such that for all $n \geq N_2$ and all $t \in X$, $|f_n(t) - f(t)| < \frac{\varepsilon}{2}$. Let $n > \max\{N_1, N_2\}$ .Then,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(b) The converse is: Suppose that $f_n \to f$ pointwise. If for every convergent sequence $(x_n) \subset X$, $x_n \to x$, $\lim_{n\to\infty} f_n(x_n) = f(x)$, then $f_n \rightrightarrows f$ on X.

A counterexample should construct f_n such that $f_n \to f$ on X pointwise and $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every convergent sequence $(x_n) \subset X$, $x_n \to x$, but the convergence of f_n to f is not uniform.

Counterexample: Take $X = \mathbb{R}$,

$$f_n(t) = \begin{cases} 1 - (t-n)^2 \text{ if } |t-n| < 1, \\ 0 \text{ otherwise.} \end{cases}$$

Then f_n are continuous on $\mathbb{R}$, and $f_n(t) = 0$ for all $n > t + 1$, so $f_n(t) \to f(t)$, where $f(t) = 0$, for all $t \in \mathbb{R}$. The convergence is not uniform because $f_n(n) = 1$, so $\sup_{t\in\mathbb{R}} |f_n(t) - f(t)| \geq 1$, for all n. Suppose $x_n \to x$. From the definition of limit and from the Archimedean principle, there exists $N > x$ such that $|x_n - x| < 1$ for all $n \geq N$. It follows that $f_n(x_n) = 0$ for all $n \geq N + 1$. In particular, $f_n(x_n) \to f(x) = 0$.