University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Jan 13, 2023

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
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Part I. Work **all** of problems 1 through 4.

Problem 1. Suppose U, W are subspaces of a finite-dimensional vector space V.

(a) Show that $\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$.

(b) Let $n = \dim V$. Show that if $k < n$ then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$.

Problem 2.

(a) For each pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{C}^3$, assign a scalar $(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^* \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \boldsymbol{x}.$$

where $\boldsymbol{y}^*$ is the conjugate transpose of $\boldsymbol{y}$. Is $(\cdot, \cdot)$ an inner product on $\mathbb{C}^3$?

(b) Let V be an inner product space and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$. Prove or disprove

(i) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u} + \boldsymbol{w}\| + \|\boldsymbol{w} + \boldsymbol{v}\|$;

(ii) $|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq |\langle \boldsymbol{u}, \boldsymbol{w} \rangle| + |\langle \boldsymbol{w}, \boldsymbol{v} \rangle|$.

Problem 3. Let T be a positive operator on a complex inner product space V and S be an operator on V such that $ST = -TS$. Show that $ST = TS = 0$.

Problem 4. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$.

(a) (5 pts) Prove that T is invertible.

(b) (15 pts) Find the minimal polynomial of T^{-1}.

Part II. Work **two** of problems 5 through 8.

Problem 5. Suppose A is a normal matrix such that $A^5 = A^4$.

(a) (8 pts) Prove that A is self-adjoint.

(b) (5 pts) Give a counterexample to Part (a) if A is not normal.

(c) (7 pts) Prove or disprove that A is a projection matrix. (Recall that a matrix A is a projection matrix if $A^2 = A$.)

Problem 6. Let V be a finite-dimensional inner product space over $\mathbb{C}$. Let T be a normal operator on V. Let $\lambda \in \mathbb{C}$ and let $v \in V$ be a unit vector (i.e. $\|v\| = 1$). Prove that T has an eigenvalue λ' such that

$$\|\lambda - \lambda'\| \leq \|Tv - \lambda v\|.$$

Problem 7. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be two sets of vectors of an inner product space V of dimension n. Suppose

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle, \quad i, j = 1, 2, \ldots, n.$$

(a) Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t\}$, $t \leq n$, be a basis for span $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$. Show that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_t\}$ is a basis for span $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

(b) Show that there exists an isometry S on V such that

$$S(\boldsymbol{u}_i) = \boldsymbol{v}_i, \quad i = 1, 2, \ldots, n.$$

Problem 8. Let V be a real inner product space and P a projection operator on V, $P^2 = P$. Prove that operator $I - 2P$ is an isometry if and only if P is self-adjoint.
