PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
FEBRUARY 11, 2021

Student ID: ______________________________

- The examination consists of four parts associated with four skills and content areas summarized to students via email. We summarize this below.
  - **Part 1: Skills and Content Area.** Produce straightforward proofs based on general metric space definitions for point-set topology and continuous functions that may, for establishing certain steps in the proofs, utilize standard results from either undergraduate or graduate analysis (MATH 4310 or 5070, respectively). **Students solve both problems in this part.**
  - **Part 2: Skills and Content Area.** Produce proofs involving commonly studied sequence/function spaces such as ℓ^p (for $1 \leq p \leq \infty$) or $\mathcal{C}^k(X, Y)$ for some $k \in \mathbb{N}$. **Students solve both problems in this part.**
  - **Part 3: Skills and Content Area.** Identify the correct theorem to apply to prove a result by proper verification of the theorem's hypothesis. The focus is on major theorems spanning all content including some of the major results from undergraduate analysis. Such theorems include, but are not limited to, the intermediate value theorem, the mean value theorem, the Fundamental Theorem of Calculus, the Arzelà–Ascoli theorem, and the contraction mapping theorem. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
  - **Part 4: Skills and Content Area.** Prove results requiring definitions and/or theorems for differentiation/integration. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
- Make sure to justify your solutions/proofs by citing theorems that you use, provide counter-examples with explanations, follow proper proof-writing techniques, etc.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of each piece of paper**; number all pages throughout; and, just in case, write your student ID on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end.
- Ask the proctor if you have any questions.

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Part 1
Students should complete both problems.

(1) Let (X, d) be a metric space and $A \subset X$.

We say that $x \in X$ is a **limit point** of A if for all $\epsilon > 0$, $B_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$. Here, $A \setminus \{x\} = A \cap \{x\}^c$. The set $B_\epsilon(x)$ denotes the ball of radius ϵ centered at the point x.

(a) (10 points) Use the given definition to prove that if $A \subset X$ has a finite number of points then A has no limit points.

(b) (5 points) Give an example of (X, d) and infinite set $A \subset X$ that has no limit points. Justify your example using the given definition.

(c) (5 points) Give an example of (X, d) and infinite set $A \subset X$ such that every point of A is also a limit point of A. Justify your example using the given definition.

Part (a). The key is to use the logical negation of the definition: We say that $x \in X$ is not a limit point of A if there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$.

Proof.

Let $\{x_n\}_{n=1}^N$ denote an enumeration of the finite points of A and let $x \in X$.

There are two cases. Either $x \in A$ or $x \notin A$.

In the first case, $x = x_m$ for some $1 \leq m \leq N$. Choose this m and choose $\epsilon = \min\{d(x_n, x) : 1 \leq n \leq N, n \neq m\} > 0$ (this is greater than zero because it is a minimum of a finite set of positive numbers).

By construction, $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$.

In the second case, choose $\epsilon = \min\{d(x_n, x) : 1 \leq n \leq N\} > 0$ (this is again greater than zero because it is a minimum of a finite set of positive numbers).

By construction, $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$.

$\square$

Part (b). Consider $\mathbb{R}$ with the usual metric and let $A = \mathbb{Z}$. Let $x \in \mathbb{R}$. If $x \in A$, choosing $\epsilon = 1$ gives $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$. If $x \notin A$, then $x \in (n, n+1)$ for some integer n and choosing $\epsilon = \min\{|x - n|, |n + 1 - x|\}$ gives $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$.

Part (c). Consider $\mathbb{R}$ with the usual metric and let $A = (0, 1)$. Let $x \in A$, then for any $\epsilon > 0$, $B_\epsilon(x) \cap (A \setminus \{x\})$ defines a sub-interval of $(0, 1)$ containing an infinite number of points from A, i.e., $B_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces.

We say that $f : A \subseteq X \to Y$ is **uniformly continuous on** A if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, z \in A$ with $d_X(x, z) < \delta$ implies $d_Y(f(x), f(z)) < \epsilon$.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $\lim_{x\to-\infty} f(x) = \alpha \in \mathbb{R}$ and $\lim_{x\to\infty} f(x) = \beta \in \mathbb{R}$. Use the given definition of uniform continuity, appropriately adapted to $\mathbb{R}$, to prove that f is uniformly continuous on $\mathbb{R}$.

Note: Appropriately adapted here simply means we use $\mathbb{R}$ with the usual metric instead of (X, d_X) and (Y, d_Y) in the definition. Also, the set A in the definition is clearly $\mathbb{R}$ in the problem statement.

Proof.

Let $\epsilon > 0$.

Since the limits at $\pm\infty$ exist and are finite, choose $a < 0$ and $b > 0$ such that $x \le a$ implies $|f(x) - \alpha| < \epsilon/2$ and $x \ge b$ implies $|f(x) - \beta| < \epsilon/2$. Since f is continuous on $\mathbb{R}$, a standard result is that f is uniformly continuous on $[a - 1, b + 1]$. Choose $\delta_1 > 0$ such that $x, z \in [a - 1, b + 1]$ with $|x - z| < \delta_1$ implies $|f(x) - f(z)| < \epsilon/2$. Choose $\delta = \min\{\delta_1, 1\}$.

Let $x, z \in \mathbb{R}$ such that $|x - z| < \delta$. Without loss of generality, assume $x < z$.

There are three (non-mutually exclusive) cases: (i) $a - 1 < x < z < b + 1$, (ii) $x < z < a$, or (iii) $b < x < z$.

Case (i): $|x - z| < \delta \le \delta_1$ implies $|f(x) - f(z)| < \epsilon/2 < \epsilon$.

Case (ii): $|f(x) - f(z)| = |f(x) - \alpha + \alpha - f(z)| \le |f(x) - \alpha| + |\alpha - f(z)|$ (by the triangle inequality), and $|f(x) - \alpha| + |\alpha - f(z)| < \epsilon/2 + \epsilon/2 = \epsilon$ since $x < z < a$.

Case (iii): $|f(x) - f(z)| = |f(x) - \beta + \beta - f(z)| \le |f(x) - \beta| + |\beta - f(z)|$ (by the triangle inequality), and $|f(x) - \beta| + |\beta - f(z)| < \epsilon/2 + \epsilon/2 = \epsilon$ since $b < x < z$. $\square$

Part 2

Students should complete both problems.

(3) Let $\mathcal{F}$ be a family of functions from a metric space (X, d_X) to a metric space (Y, d_Y). The family $\mathcal{F}$ is **equicontinuous** if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that for $z \in X$ with $d_X(x, z) < \delta$ implies $d_Y(f(x), f(z)) < \epsilon$ for all $f \in \mathcal{F}$.

Let (X, d_X) and (Y, d_Y) be metric spaces and $\mathcal{C}(X, Y)$ be the space of all continuous functions from X to Y. Given a sequence of functions $(f_n) \subset \mathcal{C}(X, Y)$, we say that $f_n \to f$ **uniformly** if

$$\sup_{x \in X} d_Y(f(x), f_n(x)) \to 0.$$

(a) (10 points) Prove that if $(f_n) \subset \mathcal{C}(X, Y)$ and $f_n \to f$ uniformly, then $\{f_n\}_{n \in \mathbb{N}} \cup \{f\}$ is equicontinuous using the given definition.

(b) (10 points) Show the necessity of uniform convergence by providing an example of (X, d_X), (Y, d_Y) and $(f_n) \subset \mathcal{C}(X, Y)$ that converges pointwise but not uniformly to $f : X \to Y$ such that $\{f_n\}_{n \in \mathbb{N}}$ is not equicontinuous. Make sure to justify your example.

Part (a). We provide a proof that is not based on any other prior results. A slightly shorter proof is possible by using the standard result to establish the continuity of f first and incorporating the associated $\delta_f > 0$ into the construction of δ.

Proof.
Let $x \in X$ and $\epsilon > 0$.

For each $n \in \mathbb{N}$, $f_n : X \to Y$ is continuous, so choose $\delta_n > 0$ such that for $z \in X$ with $d_X(x, z) < \delta_n$ implies $d_Y(f_n(x), f_n(z)) < \epsilon/6$. Since $f_n \to f$ uniformly, choose N such that $n \geq N$ implies $\sup_{x \in X} d_Y(f(x), f_n(x)) < \epsilon/6$. Choose $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_N\} > 0$.

Let $z \in X$ such that $d_X(x, z) < \delta$.

Let $g \in \{f_n\}_{n \in \mathbb{N}} \cup \{f\}$. There are three cases: (i) $g = f$, (ii) $g = f_n$ for some $n \geq N$, or (iii) $g = f_n$ for some $n < N$.

For case (i), the triangle inequality implies

$$d_Y(f(x), f(z)) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(z), f_N(z)) + d_Y(f_N(z), f(z)),$$

and the first and third terms on the right-hand side are less than $\epsilon/6$ because of the choice of N whereas the middle term on the right-hand side is less than $\epsilon/6$ by the choice of δ, so $d_Y(f(x), f(z)) < \epsilon/2 < \epsilon$.

For case (ii), the triangle inequality implies

$$d_Y(f_n(x), f_n(z)) \leq d_Y(f_n(x), f(x)) + d_Y(f(x), f(z)) + d_Y(f(z), f_n(z)),$$

and the first and third terms on the right-hand side are less than $\epsilon/6$ because of $n \geq N$ whereas the middle term on the right-hand side is less than $\epsilon/2$ by the choice of δ and the conclusion of case (i) above, so $d_Y(f_n(x), f_n(z)) < 5\epsilon/6 < \epsilon$.

For case (iii), $\delta \leq \delta_n$ implies $d_Y(f_n(x), f_n(z)) < \epsilon/6 < \epsilon$.

$\square$

Part (b). (2 points for a correct example, 3 points for justifying convergence is not uniform, 5 points for justifying not equicontinuous). Let $X = Y = [0, 1]$ with the usual metric and for each $n \in \mathbb{N}$,

$$f_n(x) = x^n$$

Clearly, $f_n(x) \to 0$ for each $x \in [0, 1)$ and $f_n(1) \to 1$. Defining f as this pointwise limit, $\sup_{x\in[0,1]} |x^n - f(x)| = \sup_{x\in[0,1)} x^n = 1$ for each n, so the convergence is not uniform. Choose $x = 1$, $\epsilon = 0.5$ and let $\delta > 0$, then choose $y \in (1-\delta, 1)$ and choose n such that $y^n < \epsilon$. This implies $|f_n(x) - f_n(y)| \geq \epsilon$, so $\{f_n\}_{n\in\mathbb{N}}$ is not equicontinuous.

(4) If (X, d_X) is a metric space and $S \subseteq X$, then we say that S is **dense** in X if $x \in X$ implies either $x \in S$ or x is a limit point of S. A metric space is called **separable** if it contains a countable dense subset. Examples of uncountable but separable metric spaces include $\mathbb{R}^k$ for any finite positive integer k.

Let (ℓ^∞, d) denote the usual metric space defined by bounded real-valued sequences, i.e., for all $(\xi_i) \in \ell^\infty$, then there exists $c \geq 0$ such that $\sup_{i\in\mathbb{N}} |\xi_i| \leq c$, and for each $x = (\xi_i), y = (\eta_i) \in \ell^\infty$,

$$d(x, y) := \sup_{i\in\mathbb{N}} |\xi_i - \eta_i|.$$

Let $A := \{(\xi_i) \in \ell^\infty : \xi_i = 0 \text{ or } 1 \ \forall i \in \mathbb{N}\}$.

(a) (5 points) Prove that A is uncountable.

(b) (5 points) Prove that A has no limit points using the definition of limit points from Problem (1) of this exam.

(c) (10 points) Prove that ℓ^∞ is *not* separable.

Part (a). This requires Cantor's diagonalization argument.

Part (b). Let $x \in \ell^\infty$. Choose $\epsilon = 1/2$, then $B_\epsilon(x) \cap A$ can contain at most one point of A by construction since each point in A is a distance of 1 away from all other points in A. Thus, by construction $B_\epsilon(x) \cap (A \setminus \{x\}) = \emptyset$. $\square$

Part (c). Let $S \subset \ell^\infty$ be countable. By parts (a) and (b), $\{B_{1/2}(x)\}_{x\in S}$ contains at most a countably infinite number of points from the uncountably infinite set A. Thus, there exists a point $x \in A \subset \ell^\infty$ such that $x \notin B_{1/2}(x) \cap S$ meaning $x \notin S$ nor is x a limit point of S. $\square$

Part 3

Students should choose one of the following two problems to complete.

(5) Let $f : [0,1] \to (0,1)$ be continuously differentiable and $\max_{0\le x\le 1} |f'(x)| \le 1-\epsilon$ for some $\epsilon > 0$. Prove that f has exactly one fixed point.

Proof.

Since $f : [0,1] \to (0,1)$, we have that $f(0) > 0$ and $f(1) < 1$.

The function $g(x) = f(x) - x$ is then continuous with $g(0) = f(0) - 0 = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.

Thus, by the Intermediate Value Theorem, $g(x)$ has a zero in the interval $(0,1)$. Choose $c \in (0,1)$ such that $g(c) = f(c) - c = 0$, which implies $f(c) = c$.

To show uniqueness of c, note that $\max_{0\le x\le 1} |f'(x)| \le 1-\epsilon < 1$ implies that $g'(x) = f'(x) - 1 < 0$, so g is strictly decreasing on $(0,1)$ and cannot have more than one zero. $\square$

(6) Let $K \in \mathcal{C}([0,1] \times [0,1])$. For $f \in \mathcal{C}([0,1])$, let

$$Tf(x) := \int_0^1 K(x,y)f(y)\,dy.$$

(a) (10 points) Prove that $Tf \in \mathcal{C}([0,1])$.

(b) (10 points) Let $\mathcal{C}([0,1])$ be equipped with the sup-norm metric. Show that $\mathcal{F} := \{Tf : \|f\|_\infty \le 1\}$ is precompact in $\mathcal{C}([0,1])$. Recall that a set is called precompact if its closure is compact.

Part (a).

Proof.

Let $x \in [0,1]$ and $\epsilon > 0$.

Since $[0,1]$ is compact and f is continuous, f is bounded by a standard result, so choose $M \ge 0$ such that $\sup_{x\in[0,1]} |f(x)| \le M$. Since $K \in \mathcal{C}([0,1]\times[0,1])$, choose $\delta > 0$ such that $\|(x_1,y_1)-(x_2,y_2)\| < \delta$ (where $\|\cdot\|$ denotes the usual 2-norm on $\mathbb{R}^2$) implies $|K(x_1,y_1) - K(x_2,y_2)| < \epsilon/(M+1)$.

Let $z \in [0,1]$ such that $|x-z| < \delta$, then

$$\begin{aligned}
|Tf(x) - Tf(z)| &= \left| \int_0^1 K(x,y)f(y)\,dy - \int_0^1 K(z,y)f(y)\,dy \right| \\
&= \left| \int_0^1 (K(x,y) - K(z,y))\,f(y)\,dy \right| \quad \text{(linearity of integral)} \\
&\le \int_0^1 |K(x,y) - K(z,y)|\,|f(y)|\;dy \quad \text{(standard integral result)} \\
&< \int_0^1 \epsilon \\
&= \epsilon.
\end{aligned}$$

□

Part (b).

Proof.

Let $Tf \in \mathcal{F}$, then by standard integral results $\|f(x)\|_\infty \le 1$ implies

$$|Tf(x)| = \left| \int_0^1 K(x,y)f(y)\,dy \right| \le \int_0^1 |K(x,y)|\;dy.$$

Since $K(x, y)$ is continuous on compact $[0, 1] \times [0, 1]$, K is bounded by a standard result, which implies $\mathcal{F}$ is uniformly bounded by the above inequality.

In the proof of Part (a), the δ was chosen independent of f, so $\mathcal{F}$ is equicontinuous.

It follows that the Arzelá-Ascoli theorem applies, which finishes the proof.

$\square$

Part 4

Students should choose one of the following two problems to complete.

(7) Suppose that for every $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that $\sup_{x\in\mathbb{R}, n\in\mathbb{N}} |f_n'(x)| = M < \infty$. Furthermore, suppose that (f_n) converges for all $x \in \mathbb{R}$ and define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$.

(a) (10 points) Show that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded on each fixed interval $[a, b]$.

(b) (10 points) Show that f is continuous on $\mathbb{R}$.

Part (a). By the Fundamental Theorem of Calculus, for each n and any $x \in \mathbb{R}$,

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\, dt.$$

Since $f_n(a) \to f(a)$, a standard result implies that the sequence of numbers $(f_n(a))$ is bounded. Choose $M \geq 0$ such that $\sup_{n\in\mathbb{N}} |f_n(a)| \leq M$. Then, for each n, the triangle inequality implies

$$|f_n(x)| \leq |f_n(a)| + \left| \int_a^x f_n'(t)\, dt \right| \leq N + M\,|x - a|\,.$$

Thus, if $x \in [a, b]$, then $|f_n(x)| \leq N + M\,|b - a|$. $\square$.

Part (b). Using the Fundamental Theorem of Calculus as in part (a), we have that for any $x, y \in \mathbb{R}$ with $x < y$,

$$\begin{aligned}
|f(y) - f(x)| &= \lim_{n\to\infty} |f_n(y) - f_n(x)| \\
&= \lim_{n\to\infty} \left| \left[f_n(a) + \int_a^y f_n'(t)\, dt \right] - \left[f_n(a) + \int_a^x f_n'(t)\, dt \right] \right| \\
&= \lim_{n\to\infty} \left| \int_x^y f_n'(t)\, dt \right| \\
&\leq \limsup_{n\to\infty} \int_x^y |f_n'(t)\, dt| \\
&\leq M\,|y - x|\,.
\end{aligned}$$

This shows f is Lipschitz continuous from which uniform and pointwise continuity follow from standard results.

(8) The Fundamental Theorem of Calculus states that if f is a real-valued continuous function on $[a, b]$ that is differentiable on (a, b), and if f' is Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Prove the above version of the Fundamental Theorem of Calculus.

Proof. Found in any standard advanced calculus textbook. □