University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
Aug 05, 2022

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1.

1. Let v_1, v_2, v_3 be linear dependent, and v_2, v_3, v_4 linear independent.
   (a) Show that v_1 is a linear combination of v_2 and v_3.
   (b) Show that v_4 is not a linear combination of v_1, v_2, v_3.
2. Find 10 vectors in $\mathbb{R}^3$ so that any three of them form a basis. Justify your answer.

Solution

1. (a) Let $a_1v_1 + a_2v_2 + a_3v_3 = 0$, where we do not have $a_1 = a_2 = a_3 = 0$. Such a_1, a_2, a_3 exist since $\{v_1, v_2, v_3\}$ is linearly dependent. If $a_1 = 0$, then $a_2v_2 + a_3v_3 = 0$ and thus $a_2 = a_3 = 0$, since $\{v_2, v_3, v_4\}$ and thus $\{v_2, v_3\}$ are linearly independent, a contradiction. Thus, $a_1 \neq 0$ and thus a_1^{-1} exists. This implies that $v_1 = -a_1^{-1}a_2v_2 - a_1^{-1}a_3v_3$, the desired linear combination.
   (b) Assume that $v_4 = b_1v_1 + b_2v_2 + b_3v_3$. Then by (a), $v_4 = (b_2 - b_1a_1^{-1}a_2)v_2 + (b_3 - b_1a_1^{-1}a_3)v_3$, contradicting that $\{v_2, v_3, v_4\}$ is linearly independent.
2. One such set of vectors is the set of $(1, i, i^2)$ for $i = 1, 2, \ldots, 10$. To see this, for $i < j < k$ consider the (Vandermonde) matrix $A = \begin{pmatrix} 1 & i & i^2 \\ 1 & j & j^2 \\ 1 & k & k^2 \end{pmatrix}$ with

   $$\det A = jk^2 + ki^2 + ij^2 - kj^2 - ji^2 - ik^2 = (k-i)(k-j)(j-i) > 0.$$

   Alternative: Any set of vectors $(1, i, f(i))$ with $f > 0$ growing fast enough will work, albeit the argument will be less elegant.

   Again, take $1 \leq i < j < k \leq 10$, and assume that

   $$(1, i, f(i)) = a(1, j, f(j)) + b(1, k, f(k)).$$

   Then $1 = a + b$ and $i = aj + bk$, so $a = \frac{i-k}{j-k}$ and $b = \frac{i-j}{k-j}$. Further,

   $$f(i) = af(j) + bf(k) = \frac{i-k}{j-k}f(j) + \frac{i-j}{k-j}f(k).$$

   If $f(k) > \frac{k-i}{j-i}f(j)$, then this implies that $f(i) < 0$, a contradiction.
   Since $\frac{k-i}{j-i} \leq k - i \leq 9$, using $f(i) = 10^i$ works for this purpose.

Problem 2.

Let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^p$. The matrix norm induced by $\|\cdot\|$ is defined by

$$\|P\| = \max_{x\neq 0} \frac{\|Px\|}{\|x\|}$$

for each $p \times p$ real matrix P.

1. Prove that $\|\cdot\|$ is a norm on the vector space of real $p \times p$ matrices.
2. Let A, B be $p \times p$ real matrices. Show that
$$\|AB\| \leq \|A\|\|B\|.$$
3. Let P be a $p \times p$ real matrix. Suppose that $\|P\| < 1$. Prove that $I+P$ is nonsingular and that
$$\frac{1}{1+\|P\|} \leq \|(I+P)^{-1}\| \leq \frac{1}{1-\|P\|}.$$

Solution

1. We need to verify that the induced norm satisfies the three properties of norms: 1) $\|P\| > 0$ for $P \neq 0$; 2) for any scalar α and matrix P, $\|\alpha P\| = |\alpha|\|P\|$ and 3) for any two matrix P and Q, $\|P\| + \|Q\| \leq \|P\| + \|Q\|$.

   1) Since $\|\cdot\|$ is a vector norm, $\|Px\| \geq 0$ for all P and x. Thus, the right hand side in the definition above is always nonnegative, so $\|P\| \geq 0$. Moreover, if $P \neq 0$, it has rank ≥ 1; thus, we can find $\bar{x} \in \mathbb{R}^p$ such that $P\bar{x} \neq 0$. But then $\|P\| \geq \frac{\|P\bar{x}\|}{\|x\|} > 0$. Thus, $\|P\| > 0$ for all $P \neq 0$.

   2) For any scalar α we have
$$\|\alpha P\| = \max_{x\neq 0} \frac{\|\alpha Px\|}{\|x\|} = \max_{x\neq 0} \frac{|\alpha|\|Px\|}{\|x\|} = |\alpha| \max_{x\neq 0} \frac{\|Px\|}{\|x\|} = |\alpha|\|P\|.$$

   3) For two matrices P and Q, we have
$$\begin{aligned}\|P+Q\| = \max_{x\neq 0} \frac{\|(P+Q)x\|}{\|x\|} &\leq \max_{x\neq 0} \frac{\|Px\| + \|Qx\|}{\|x\|} \\ &\leq \max_{x\neq 0} \frac{\|Px\|}{\|x\|} + \max_{y\neq 0} \frac{\|Qy\|}{\|y\|} = \|P\| + \|Q\|\end{aligned}$$

2.

$$\|AB\| = \max_{x\neq 0} \frac{\|ABx\|}{\|x\|} \leq \max_{x\neq 0} \frac{\|A\|\|Bx\|}{\|x\|} = \|A\| \max_{x\neq 0} \frac{\|Bx\|}{\|x\|} = \|A\|\|B\|$$

3. Suppose x is a solution to the equation $(I+P)x = 0$. Then $x = -Px$, so

$$\|x\| = \|-Px\| \leq \|P\|\|x\|.$$

Since $\|P\| < 1$, this implies that $x = 0$. (Otherwise, we get the contradiction $\|x\| < \|x\|$). Thus, the only solution to $(I+P)x = 0$ is the trivial solution $x = 0$, so $I + P$ is nonsingular.

Let $B = (I+P)^{-1}$. Then $I = B(I+P)$. Thus,

$$1 = \|I\| = \|B(I+P)\| \leq \|B\|\|I+P\| \leq \|B\|(1+\|P\|).$$

Thus,

$$\frac{1}{1+\|P\|} \leq \|B\| = \|(I+P)^{-1}\|.$$

To get the second inequality, observe that $I = B + BP$, so $B = I - BP$. Thus,

$$\|B\| = \|I - BP\| \leq 1 + \|BP\| \leq 1 + \|B\|\|P\|.$$

Hence, $\|B\|(1-\|P\|) \leq 1$ and $\|B\| \leq \frac{1}{1-\|P\|}$.

Problem 3. Let A be a Hermitian $n \times n$ complex matrix. Show that if $< A\boldsymbol{v}, \boldsymbol{v} >\geq 0$ for all $\boldsymbol{v} \in \mathbb{C}^n$ then there exists an $n \times n$ matrix T such that $A = T^*T$ (Here T^* is the conjugate transpose of T).

Solution Due to the spectral theorem, an orthonormal basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ exists such that

$$A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n = \lambda_n \boldsymbol{v}_n$$

Note that $\lambda_i, 1 \leq i \leq n$, are real, since A is Hermitian.

For each i, we further have

$$< A\boldsymbol{v}_i, \boldsymbol{v}_i >=< \lambda \boldsymbol{v}_i, \boldsymbol{v}_i >= \lambda_i < \boldsymbol{v}_i, \boldsymbol{v}_i >$$

which is nonnegative. So $\lambda_i, 1 \leq 1 \leq n$, must be nonnegative.

Hence, it makes sense to define the linear map (matrix) T such that

$$T\boldsymbol{v}_i = \sqrt{\lambda_i}\boldsymbol{v}_i$$

Such matrix exists and is unique, since both $\boldsymbol{v}_i$ and $\sqrt{\lambda_i}\boldsymbol{v}_i$ are bases of $\mathbb{C}^n$. Note that the matrix of T, $\mathcal{M}(T)$, with respect to the basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is diagonal, with $\sqrt{\lambda_i}$'s on the diagonal. Therefore $\mathcal{M}(T^*)$ is also diagonal, with $\overline{\sqrt{\lambda_i}} = \sqrt{\lambda_i}$ on the diagonal. So we have

$$T^*(\boldsymbol{v}_i) = \sqrt{\lambda_i}\boldsymbol{v}_i$$

Finally, we have

$$A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i = \sqrt{\lambda_i}(\sqrt{\lambda_i}\boldsymbol{v}_i) = \sqrt{\lambda_i}T^*\boldsymbol{v}_i = T^*(\sqrt{\lambda_i}\boldsymbol{v}_i) = T^*T\boldsymbol{v}_i$$

This means A and TT^* agree on the basis $\boldsymbol{v}_i$, and it implies $A = TT^*$.

Problem 4.

Let n be an integer. Let A be the n-by-n matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}.$$

So A has 1's everywhere but 0's on the diagonal. Or in other words, for all $1 \leq i, j \leq n$, $a_{ij} = 1$ if $i \neq j$ and $a_{ij} = 0$ if $i = j$.

Give the determinant of A as a function of n.

Solution

$$\det(A) = \begin{vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 0 \end{vmatrix}.$$

We substract the first row to all other rows. So we perform the following elementary row operations: $L_2 \to L_2 - L_1$, then $L_3 \to L_3 - L_1$, then $L_4 \to L_4 - L_1$, etc. These operations do not change the determinant. and we get

$$\det(A) = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & -1 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}.$$

We add the second to the last columns to the first column. So we perform the following elementary column operations: $C_1 \to C_1 + C_2$, then $C_1 \to C_1 + C_3$, then $C_1 \to C_1 + C_4$, etc. These operations do not change the determinant. We get

$$\det(A) = \begin{vmatrix} (n-1) & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}.$$

We can now conclude that

$$\boxed{\det(A) = (-1)^{n-1}(n-1).}$$

Solution 2: (Alternate)

• Let $x^{(1)}$ be the vector with all ones. We have that $Ax^{(1)} = (n-1)x^{(1)}$. So $x^{(1)}$ is an eigenvector of eigenvalue $(n-1)$ of A.

• For $2 \leq k \leq n$, let $x^{(k)}$ be the vector with $x^{(k)}_{k-1} = 1$ and $x^{(k)}_k = -1$ and $x^{(k)}_j = 0$ if $(j \neq k-1$ and $j \neq k)$. We have that $Ax^{(k)} = -x^{(k)}$. So $x^{(k)}$ is an eigenvector of eigenvalue -1 of A.

• The n vectors $x^{(n)}$ look like

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad x^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, x^{(n)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 1 \end{pmatrix}.$$

• The set of vectors $\left(x^{(k)}\right)_{2\leq k\leq n}$ is linearly independent based on its zero structure. So the eigenspace associated with the eigenvalue -1 is of dimension at least $n-1$. And the dimension is actually exactly $n-1$ because we have a second eigenvalue distinct from -1, namely $n-1$.

• In all, A has the eignvalue $n-1$ with (geometric and algebraic) multiplicity 1. and the eignvalue -1 with (geometric and algebraic) multiplicity $n-1$.

• Note relevant for our proof: A is diagonalizable.

• The determinant is the product of the eigenvalues with their algebraic multiplicity, therefore

$$\boxed{\det(A) = (-1)^{n-1}(n-1).}$$

Part II. Work **two** of problems 5 through 8.

Problem 5.

We assume that the following result is true.

Let $A = (a_{ij})$ in $\mathcal{M}_n(\mathbb{R})$ with $a_{ii} = 0$ for all i and $|a_{ij}| = 1$ for all $i \neq j$. (So that, for $i \neq j$, a_{ij} is either 1 or -1.) If n is even, then A is invertible.

Let $n \geq 1$. Let us consider $2n+1$ stones. We assume that each subset of $2n$ stones can be divided in 2 sets of n stones such that the two sets have same weight. Show that all the stones have the same weight.

Solution • We number the stones from 1 to $2n+1$. For $1 \leq i \leq 2n+1$, let w_i be the weight of stone i. Let w be the vectors of the weights.

• Let $1 \leq i \leq 2n+1$. We consider the set of $2n$ stones made of the $2n+1$ stones without stone i. Per assumption, we can partition this set in 2 sets of n stones such that the two sets have same weight. Let us call these two sets $\mathcal{S}_1^{(i)}$ and $\mathcal{S}_2^{(i)}$. We have that the cardinal of $\mathcal{S}_1^{(i)}$ and the cardinal of $\mathcal{S}_2^{(i)}$ is n. The fact that the sets have the same weight reads

$$\sum_{k \in \mathcal{S}_1^{(i)}} w_k = \sum_{k \in \mathcal{S}_2^{(i)}} w_k. \tag{1}$$

So we get an equation of the type:

$$\sum_{1 \leq j \leq 2n+1} a_{ij} w_j = 0, \tag{2}$$

where

$$|a_{ij}| = 1, \text{ for } (1 \leq j \leq 2n+1, j \neq i); \quad \text{ and } a_{ij} = 0, \text{ for } j = i; \tag{3}$$

$$\sum_{1 \leq j \leq 2n+1} a_{ij} = 0 \tag{4}$$

Equation (4) comes from the fact that, for $1 \leq j \leq 2n+1$, (a) one a_{ij} is 0, namely a_{ii}, (b) n a_{ij} are 1, and (c) n a_{ij} are -1, so that the sum of all is zero. The fact that ((n a_{ij} are

1) and (n a_{ij} are -1)) comes from the fact that the two sets $\mathcal{S}_1^{(i)}$ and $\mathcal{S}_2^{(i)}$ have cardinals n.

• Let $A = (a_{ij})$ in $\mathcal{M}_{2n+1}(\mathbb{R})$.

• Let x the vector of size $2n+1$ with all ones. Equation (4) implies that $Ax = 0$. So that $x \in \text{Null}(A)$ and $\dim(\text{Null}(A)) \geq 1$.

• Comment: $\dim(\text{Null}(A)) \geq 1$ means that A is not invertible. "A is not invertible" is not a contradiction of the "result" because the "result" is valid when the size of A is even. Here the size of A is $2n+1$ and therefore odd.

• Equation (2) implies that $Aw = 0$. So that $w \in \text{Null}(A)$.

• Time to use the "result". We note that the $(2n)$-by-$(2n)$ submatrix $A(1:2n, 1:2n)$ has the property of the "result". Therefore $A(1:2n, 1:2n)$ is invertible. Therefore, because A has a $(2n)$-by-$(2n)$ invertible submatrix, we can say that $\text{Rank}(A) \geq 2n$.

• We have (a) $\dim(\text{Null}(A)) \geq 1$, (b) $\text{Rank}(A) \geq 2n$, and (c), by rank theorem, $\dim(\text{Null}(A)) + \text{Rank}(A) = 2n+1$, therefore we conclude that $\dim(\text{Null}(A)) = 1$ and $\text{Rank}(A) = 2n$.

• We know that (a) $\dim(\text{Null}(A)) = 1$, (v) $w \in \text{Null}(A)$, and (c) $x \in \text{Null}(A)$. It must therefore be that w and x are colinear vectors. In other words, since x is not zero, there exists $\alpha \in \mathbb{R}$ such that $w = \alpha x$.

• There exists $\alpha \in \mathbb{R}$ such that $w = \alpha x$. But x is the vector of all ones. So this means that, for $1 \leq i \leq 2n+1$, $w_i = \alpha$.

All $2n+1$ stones weigh the same.

Comment:

We can prove the result given as a hint in this problem (Problem #5)

Let $A = (a_{ij})$ in $\mathcal{M}_n(\mathbb{R})$ with $a_{ii} = 0$ for all i and $|a_{ij}| = 1$ for all $i \neq j$. (So that, for $i \neq j$, a_{ij} is either 1 or -1.) If n is even, then A is invertible.

with a result related to Problem #4. Let us explain.

Actually the assumptions on matrix A can only be that a_{ij}, for all $i \neq j$, are odd and a_{ii} are even. (Which is a larger class of matrices than a_{ij}, for all $i \neq j$, are ± 1 and a_{ii} are zero.)

We can mod 2 the matrix with the ± 1 and the zeros and then we obtain the matrix of Problem #4, a matrix with all ones but on the diagonal, the diagonal is zero. In Problem #4, we proved that the determinant of this matrix is $(-1)^{n-1}(n-1)$. Therefore if n is even, we see that the determinant is odd. And so the determinant mod 2 is 1. All in all, we can prove that the determinant of the initial matrix (with the ± 1) is odd, (if n is even,) and therefore the initial matrix is invertible.

In this problem we are going back and forth from the field $\mathbb{R}$ to the field GF(2). We use the fact that, if we have A over $\mathbb{R}$, then the modulo 2 of the determinant of A is the determinant computed in GF(2) of A modulo 2. This fact is easily determined by realizing that the determinant is a product and addition of numbers and we know that the modulo 2 of an addition of two real numbers is the same as the addition in GF(2) of the modulo 2 of these two real numbers. Similar for multiplication.

Going from the field $\mathbb{R}$ to the field GF(2) is convenient to remove all the ± 1. They all become ones. And we end up computing the parity of the determinant using GF(2). Keeping the ± 1 would be impractical.

Another way to view the problem (which is the excatly same thing as working in GF(2)) is to write each entry as "odd" or "even" and do the determinant computation (as in Problem #4) by doing "odd"+"odd" = "even", "odd"+"even" = "odd", "odd"*"odd" = "odd", "odd"*"even" = "even", etc. Which is exactly what is going on in GF(2) arithmetic. This might easier to understand. That way we can prove that if n is even, the determinant is odd. If n is odd, the determinant is even.

If we can prove that the determinant is odd, then we proved that the matrix invertible. If we can prove that the determinant is even, then we cannot conclude anything with respect of invertibility of the matrix.

In our case, if n is even, we can prove that the determinant is odd, so we proved that the matrix invertible.

If n is odd, we can prove that the determinant is even, then we cannot conclude anything with respect of invertibility of the matrix. Indeed, in the n-is-odd case, there are cases (of combination of ± 1) when the matrix is invertible. There are cases when the matrix is not invertible.

As explained the result below is also true and follows the same proof:

Let $A = (a_{ij})$ in $\mathcal{M}_n(\mathbb{R})$ with a_{ii} even for all i and a_{ij} is odd integer for all $i \neq j$. If n is even integer, then A is invertible.

Problem 6.

Let $A \in \mathcal{M}_n(\mathbb{C})$, and λ be an eigenvalue of A.

1. Show that λ^r is an eigenvalue of A^r.
2. Provide an example showing that the geometric multiplicity of λ^r as an eigenvalue of A^r may be strictly higher than the geometric multiplicity of λ as an eigenvalue of A.
3. Show that $A^\top$ has the same eigenvalues as A.
4. Show: If A is orthogonal, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

Solution

1. Let $v \neq 0$ such that $Av = \lambda v$. Then $A^r v = A^{r-1}Av = A^{r-1}(\lambda v) = \lambda A^{r-1}v = \lambda^2 A^{r-2}v = \ldots = \lambda^r v$, so v is eigenvector of A^r with eigenvalue λ^r.
2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so A has eingenvalues 1 and -1 (eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively), each with multiplicity 1. But $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a matrix with eigenvalue 1 with multiplicity 2.
3. Note that $(A^k)^\top = (A^\top)^k$. Thus, any polynomial in A evaluates as singular if and only if it is singular in $A^\top$. This is true for the characteristic polynomial for all eigenvalues, so the eigenvalues are the same.
4. If A is orthogonal, then $A^{-1} = A^\top$. For an eigenvector v with $Av = \lambda v$ (note that $\lambda \neq 0$ for invertible matrices), we have $v = A^{-1}Av = A^\top(\lambda v) = \lambda A^\top v$, so $A^\top v = \frac{1}{\lambda}v$, and $\frac{1}{\lambda}$ is eigenvalue of $A^\top$, and thus by (c) also eigenvalue of A.

Problem 7. Let T be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(T) = T^4 + 2T^3 - 2T - I = 0$, where I is the identity operator on V. Suppose that $|\text{trace}(T)| = 2$ and that $\dim \text{range}\,(T + I) = 2$. Give a Jordan canonical form of T.

Solution First, notice $x = 1$ is a solution to the polynomial $p(x) = x^4+2x^3-2x-1 = 0$. A long division of $p(x)$ by $x - 1$ shows

$$p(x) = (x - 1)(x^3 + 3x^2 + 3x + 1) = (x - 1)(x + 1)^3$$

So possible eigenvalues of T are $\lambda_1 = -1$ and $\lambda_2 = 1$.

Based on the rank-nullity theorem:

$$\dim \operatorname{null}(T + I) = 4 - \dim \operatorname{range}(T + I) = 4 - 2 = 2$$

So the geometric multiplicity of $\lambda_1 = -1$ is 2, which means there must be 2 blocks for -1. Its algebraic multiplicity is greater than or equal to 2. Combined with $\operatorname{trace}(T) = \pm 2$, the only possibility for the diagonal elements of the Jordan canonical forms is $-1, -1, -1, 1$, i.e., the algebraic multiplicity for -1 is 3. So a possible Jordan canonical form is:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Problem 8.

1. Let T be an idempotent operator on an n-dimensional vector space V; that is, $T^2 = T$, show that

   (a) $V = \operatorname{range} T \oplus \operatorname{null} T$.

   (b) $\operatorname{trace} T = \dim \operatorname{range} T$

2. Let $T_1, T_2, \ldots, T_m$ be idempotent operators on an n-dimensional vector space V. Show that if

   $$T_1 + T_2 + \cdots + T_m = I$$

   then

   $$V = \operatorname{range} T_1 \oplus \operatorname{range} T_2 \oplus \cdots \oplus \operatorname{range} T_m$$

   and

   $$T_i T_j = 0,\ i, j = 1, 2, \ldots, m,\ i \neq j$$

Solution

1. (a) First notice that $V = \operatorname{null} T + \operatorname{range} T$, since $\boldsymbol{v} \in V$ can be written as $\boldsymbol{v} = T\boldsymbol{v} + (\boldsymbol{v} - T\boldsymbol{v})$. Now we show their intersection is $\{\boldsymbol{0}\}$.
   Let $\boldsymbol{u} \in \operatorname{null} T \cap \operatorname{range} T$. Then $\boldsymbol{u} = T\tilde{\boldsymbol{u}}$, for some $\tilde{\boldsymbol{u}} \in V$. Now

$$\boldsymbol{0} = T\boldsymbol{u} = T(T\tilde{\boldsymbol{u}}) = T^2\tilde{\boldsymbol{u}} = T\tilde{\boldsymbol{u}} = \boldsymbol{u}$$

   So the intersection is indeed $\{\boldsymbol{0}\}$, and the sum must be a direct sum.

   (b) Let $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n_1}\}$ be a basis of $\operatorname{range} T$, and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n_2}\}$ be a basis of $\operatorname{null} T$. Then

$$\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n_1}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n_2}$$

   is a basis of V. Note that $\boldsymbol{u}_i = T\tilde{\boldsymbol{u}}_i, i = 1, \dots, n_1$, for some $\tilde{\boldsymbol{u}}_i \in V$. We have

$$T\boldsymbol{u}_i = T(T\tilde{\boldsymbol{u}}_i) = T^2\tilde{\boldsymbol{u}}_i = T\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i,\ i = 1, \dots, n_1$$

   and

$$T\boldsymbol{v}_i = \boldsymbol{0}$$

   So the matrix of T corresponding to this set of basis vectors is in the form of $(1, \dots, 1, 0, \dots, 0)$. The matrix has a trace of $1 + \cdots + 1 = n_1$, which is the dimension of $\operatorname{range} T$. So $\operatorname{trace} T = \dim \operatorname{range} T$.

2. For $\boldsymbol{v} \in V$, we have

$$\boldsymbol{v} = I\boldsymbol{v} = (T_1 + \cdots + T_m)\boldsymbol{v} = T_1\boldsymbol{v} + \cdots + T_m\boldsymbol{v}$$

   which means

$$V = \operatorname{range} T_1 + \cdots + \operatorname{range} T_m$$

   Now we show that

$$\dim V = \dim \operatorname{range} T_1 + \cdots + \dim \operatorname{range} T_m$$

   Recall that we just showed that

$$\dim \operatorname{range} T_i = \operatorname{trace} T_i$$

   So

$$\begin{aligned} \dim V = n = \operatorname{trace} I_n = \operatorname{trace}(T_1 + \cdots + T_m) &= \operatorname{trace} T_1 + \cdots + \operatorname{trace} T_m \\ &= \dim \operatorname{range} T_1 + \cdots + \dim \operatorname{range} T_m \end{aligned}$$

   Now we show $T_i T_j = 0$ for distinct i, j. For any $\boldsymbol{v} \in V$, we have

$$T_j\boldsymbol{v} = (\sum_{i=1}^{n} T_i)T_j\boldsymbol{v} = \sum_{i=1}^{n} T_i T_j \boldsymbol{v}$$

which is equivalent to

$$(T_j\boldsymbol{v} - T_j^2\boldsymbol{v}) - \sum_{i=1, i\neq j}^{n} T_iT_j\boldsymbol{v} = \mathbf{0}$$

Note that

$$T_j\boldsymbol{v} \in \operatorname{range} T_j,\ T_j^2\boldsymbol{v} \in \operatorname{range} T_j,\ T_iT_j\boldsymbol{v} \in \operatorname{range} T_i$$

and V is a direct sum of range T_i's. So we know each term above is $\mathbf{0}$, due to the unique way of decomposing $\mathbf{0}$. So

$$(T_iT_j)\boldsymbol{v} = \mathbf{0},\ i \neq j.$$
