University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Aug 05, 2022

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
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Part I. Work **all** of problems 1 through 4.

Problem 1.

1. Let v_1, v_2, v_3 be linear dependent, and v_2, v_3, v_4 linear independent.
   (a) Show that v_1 is a linear combination of v_2 and v_3.
   (b) Show that v_4 is not a linear combination of v_1, v_2, v_3.
2. Find 10 vectors in $\mathbb{R}^3$ so that any three of them form a basis. Justify your answer.

Problem 2.

Let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^p$. The matrix norm induced by $\|\cdot\|$ is defined by

$$\|P\| = \max_{x \neq 0} \frac{\|Px\|}{\|x\|}$$

for each $p \times p$ real matrix P.

1. Prove that $\|\cdot\|$ is a norm on the vector space of real $p \times p$ matrices.
2. Let P be a $p \times p$ real matrix. Suppose that $\|P\| < 1$. Prove that $I + P$ is nonsingular and that
$$\frac{1}{1 + \|P\|} \leq \|(I + P)^{-1}\| \leq \frac{1}{1 - \|P\|}.$$

Problem 3. Let A be a Hermitian $n \times n$ complex matrix. Show that if $< A\boldsymbol{v}, \boldsymbol{v} > \geq 0$ for all $\boldsymbol{v} \in \mathbb{C}^n$ then there exists an $n \times n$ matrix T such that $A = T^*T$ (Here T^* is the conjugate transpose of T).

Problem 4.

Let n be an integer. Let A be the n-by-n matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

So A has 1's everywhere but 0's on the diagonal. Or in other words, for all $1 \leq i, j \leq n$, $a_{ij} = 1$ if $i \neq j$ and $a_{ij} = 0$ if $i = j$.

Give the determinant of A as a function of n.

Part II. Work **two** of problems 5 through 8.

Problem 5.

We assume that the following result is true.

Let $A = (a_{ij})$ in $\mathcal{M}_n(\mathbb{R})$ with $a_{ii} = 0$ for all i and $\lvert a_{ij}\rvert = 1$ for all $i \neq j$. (So that, for $i \neq j$, a_{ij} is either 1 or -1.) If n is even, then A is invertible.

Let $n \geq 1$. Let us consider $2n+1$ stones. We assume that each subset of $2n$ stones can be divided in 2 sets of n stones such that the two sets have same weight. Show that all the stones have the same weight.

Problem 6.

Let $A \in \mathcal{M}_n(\mathbb{C})$, and λ be an eigenvalue of A.

1. Show that λ^r is an eigenvalue of A^r.
2. Provide an example showing that the multiplicity of λ^r as an eigenvalue of A^r may be strictly higher than the multiplicity of λ as an eigenvalue of A.
3. Show that $A^\top$ has the same eigenvalues as A.
4. Show: If A is orthogonal, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

Problem 7. Let T be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(T) = T^4 + 2T^3 - 2T - I = 0$, where I is the identity operator on V. Suppose that $\lvert\operatorname{trace}(T)\rvert = 2$ and that $\dim \operatorname{range}(T + I) = 2$. Give a Jordan canonical form of T.

Problem 8.

1. Let T be an idempotent operator on an n-dimensional vector space V; that is, $T^2 = T$, show that

   (a) $V = \operatorname{range} T \oplus \operatorname{null} T$.

   (b) $\operatorname{trace} T = \dim \operatorname{range} T$

2. Let $T_1, T_2, \ldots, T_m$ be idempotent operators on an n-dimensional vector space V. Show that if

$$T_1 + T_2 + \cdots + T_m = I$$

then

$$V = \operatorname{range} T_1 \oplus \operatorname{range} T_2 \oplus \cdots \oplus \operatorname{range} T_m$$

and

$$T_i T_j = 0,\ i, j = 1, 2, \ldots, m,\ i \neq j$$