University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
February 4, 2022

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1. Let V be a vector space of dimension n over a field F. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a basis of V and T be an operator on V. Prove: T is invertible if and only if $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is linearly independent.

Solution First we prove "$\Rightarrow$". Suppose T is invertible and its inverse is T^{-1}. So T^{-1} is an operator on V. Suppose $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is linearly dependent. Then $T^{-1}(T\boldsymbol{v}_1), T^{-1}(T\boldsymbol{v}_2), \ldots, T^{-1}(T\boldsymbol{v}_n)$ is linearly dependent, i.e., $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is linearly dependent. This leads to a contradiction. So $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ must be linearly dependent.

Now we prove "$\Leftarrow$". Suppose $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is linearly independent. Since V is n-dimensional, $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is a basis of V. For any $\boldsymbol{w} \in V$, there exist $k_1, k_2, \ldots, k_n$ such that

$$\boldsymbol{w} = k_1(T\boldsymbol{v}_1) + k_2(T\boldsymbol{v}_2) + \cdots + k_n(T\boldsymbol{v}_n) = T(k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n).$$

i.e., there exists $\boldsymbol{u} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n$ such that $T\boldsymbol{u} = \boldsymbol{w}$. So T is surjective.

On the other hand, if there exists $\boldsymbol{u}_1 = l_1\boldsymbol{v}_1 + l_2\boldsymbol{v}_2 + \cdots + l_n\boldsymbol{v}_n$ such that $T\boldsymbol{u}_1 = \boldsymbol{w}$, i.e.,

$$T\boldsymbol{u}_1 = T(l_1\boldsymbol{v}_1 + l_2\boldsymbol{v}_2 + \cdots + l_n\boldsymbol{v}_n) = l_1T(\boldsymbol{v}_1) + l_2T(\boldsymbol{v}_2) + \cdots + l_nT(\boldsymbol{v}_n) = k_1(T\boldsymbol{v}_1) + k_2(T\boldsymbol{v}_2) + \cdots + k_n(T\boldsymbol{v}_n)$$

Since $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is a basis,

$$l_i = k_i, \quad (i = 1, 2, \ldots, n)$$

So $\boldsymbol{u} = \boldsymbol{u}_1$ and hence T is injective. So T is invertible.

Problem 2.

1. Give an orthonormal basis for $\texttt{null}\ T$, where $T \in \mathcal{L}(\mathbb{C}^4)$ (with the standard inner product) and
$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

2. Prove or disprove:

   There exists an inner product $\langle .,. \rangle$ on $\mathbb{R}^2$ such that for every $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we have
$$\langle v, v \rangle = |x| + |y|.$$

Solution

1. Note first that $\mathcal{M}(T)$ has rank 1, so $\texttt{dim null}\ T = 3$. A vector is in $\texttt{null}\ T$ if and only if its entries sum up to 0.
$\begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{pmatrix}$ is an ONB of $\texttt{null}\ T$.

2. There is no such inner product. Otherwise, consider $x = y = 1$. Then $\langle v, v \rangle = 1 + 1 = 2$. Further, $\langle 2v, 2v \rangle = 2 + 2 = 4$. On the other hand, by bilinearity of the inner product, $\langle 2v, 2v \rangle = 4\langle v, v \rangle = 8$, a contradiction.

Problem 3. Prove or give a counterexample to each of the following statements:

1. Let $T \in \mathcal{L}(\mathbb{R}^3)$, and $\texttt{dim}\,(\texttt{null}\ T \cap \texttt{range}\ T) \geq 1$. Then T is nilpotent.
2. Let $T \in \mathcal{L}(\mathbb{R}^4)$, and $\texttt{dim}\,(\texttt{null}\ T \cap \texttt{range}\ T) \geq 2$. Then T is nilpotent.

Solution

1. Counterexample
$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
then $\texttt{null}\ T \cap \texttt{range}\ T = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $T^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0$ for all k.

2. This is true. Since $\texttt{dim}\,\texttt{range}\ T + \texttt{dim}\,\texttt{null}\ T = 4$ we have $\texttt{dim}\,\texttt{range}\ T = \texttt{dim}\,\texttt{null}\ T = \texttt{dim}\,(\texttt{null}\ T \cap \texttt{range}\ T) = 2$, and thus $\texttt{range}\ T = \texttt{null}\ T$, and therefore $T^2 = 0$.

Problem 4. Let G, O, L, Y, N, X be 6 real numbers.

Note: the letter O is not the same as the number 0. Please make sure that you see the difference between the letters O and the numbers 0.

We consider the following 6-by-6 "Go Lynx" matrix A:

$$A = \begin{pmatrix} G & O & L & Y & N & X \\ G & -O & L & -Y & N & -X \\ G & O & L & Y & -N & -X \\ G & -O & L & -Y & -N & X \\ G & O & -L & -Y & N & X \\ G & -O & -L & Y & N & -X \end{pmatrix}.$$

Compute the determinant of A. Answer needs to be a closed form algebraic formula with variables G, O, L, Y, N, X. No matrix in final answer.

Solution First we have that

$$\det(A) = \begin{vmatrix} G & O & L & Y & N & X \\ G & -O & L & -Y & N & -X \\ G & O & L & Y & -N & -X \\ G & -O & L & -Y & -N & X \\ G & O & -L & -Y & N & X \\ G & -O & -L & Y & N & -X \end{vmatrix} = GOLYNX \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix}.$$

We perfom the row operations $L_2 \leftarrow L_2 - L_1$, $L_3 \leftarrow L_3 - L_1$, $L_4 \leftarrow L_4 - L_1$, $L_5 \leftarrow L_5 - L_1$, $L_6 \leftarrow L_6 - L_1$, and get that

$$\det(A) = GOLYNX \cdot \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & -2 \end{pmatrix}.$$

We expand with first column and get that

$$\det(A) = GOLYNX \cdot \begin{pmatrix} -2 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \\ -2 & 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 & -2 \end{pmatrix}.$$

We scale each row by $-\frac{1}{2}$ and get

$$\det(A) = -32 \cdot GOLYNX \cdot \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We perfom the row operations $L_3 \leftarrow L_3 - L_1$, $L_5 \leftarrow L_5 - L_1$, and get that

$$\det(A) = -32 \cdot GOLYNX \cdot \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

We expand with first column and get that

$$\det(A) = -32 \cdot GOLYNX \cdot \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

We perfom the row operations $L_2 \leftarrow L_2 - L_1$, $L_4 \leftarrow L_4 - L_3$, and get that

$$\det(A) = -32 \cdot GOLYNX \cdot \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

We expand with third column and get that

$$\det(A) = -32 \cdot GOLYNX \cdot \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}.$$

We expand with third column and get that

$$\det(A) = 64 \cdot GOLYNX \cdot \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.$$

We expand with first column and get that

$$\det(A) = 64 \cdot GOLYNX \cdot \begin{pmatrix} -2 \end{pmatrix}.$$

Finally

$$\det(A) = -128 \cdot GOLYNX.$$

Part II. Work **two** of problems 5 through 8.

Problem 5. Let $\boldsymbol{u}$ be a unit vector in an n-dimensional inner product space V over $\mathbb{R}$. Define $T \in \mathcal{L}(V)$ as:

$$T(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{u}\rangle \boldsymbol{u}, \;\; \boldsymbol{x} \in V$$

Show that

1. T is an isometry.
2. If $A = \mathcal{M}(T)$ is a matrix representation of T, then $\texttt{det}\ A = -1$.
3. If $S \in \mathcal{L}(V)$ is an isometry with 1 as an eigenvalue, and if the eigenspace of 1 is of dimension $n-1$, then

   $$S(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{w}\rangle \boldsymbol{w}, \;\; \boldsymbol{x} \in V$$

   for some unit vector $\boldsymbol{w} \in V$.

Solution

(a) Extend $\boldsymbol{u}$ to an orthonormal basis $\{\boldsymbol{u}, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. Then

$$\begin{aligned} T(\boldsymbol{u}) &= \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{u}\rangle \boldsymbol{u} = -\boldsymbol{u}, \\ T(\boldsymbol{u}_i) &= \boldsymbol{u}_i - 2\langle \boldsymbol{u}_i, \boldsymbol{u}\rangle \boldsymbol{u} = \boldsymbol{u}_i, \;\; i = 2, \ldots, n \end{aligned}$$

Theorem 7.42 Part (d) guarantees that T is an isometry. (Note that a student pointing out a theorem in the textbook is OK.)

(b) The matrix of T under the above basis is

$$\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

Thus $\texttt{det}\ A = -1$. A matrix representation of T under any basis is similar to A, and hence the determinant remains the same.

(c) Denote the eigenspace of 1 by V_1 and suppose that

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n-1}\}$$

is a basis for V_1. Extend the basis of V_1 to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n-1}, \boldsymbol{u}_n\}$, a basis of V. WLOG, we can assume that $\boldsymbol{u}_n$ is a unit vector. Considering the matrix representation of S with respect to this basis, for the matrix to be orthogonal, the only possible case is $S(\boldsymbol{u}_n) = -\boldsymbol{u}_n$. That is, -1 is an eigenvalue of S, with $\boldsymbol{u}_n$ as a corresponding eigenvector. Hence now we have

$$S\boldsymbol{u}_i = \boldsymbol{u}_i,\ i = 1, 2, \ldots, n-1,\ \ S\boldsymbol{u}_n = -\boldsymbol{u}_n$$

Replacing $\boldsymbol{u}_n$ by $\boldsymbol{w}$, we see the basis vectors $\{\boldsymbol{u}_1, \ldots \boldsymbol{u}_{n-1}, \boldsymbol{w}\}$ satisfy $S(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{w}\rangle \boldsymbol{w}$, and thus any vector $\boldsymbol{x} \in V$ satisfies it too. So $S(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{w}\rangle \boldsymbol{w}$.

Problem 6. Let V be an n-dimensional vector space, and let $T_1, \dots, T_{n+1} \in \mathcal{L}(V)$ such that

(i) $T_i T_j = T_j T_i$ for every $1 \leq i \leq j \leq n+1$ (the operators commute), and

(ii) $T_1 T_2 \dots T_{n+1} = 0$.

1. (15 points)
   Show that there exists some k such that $T_1 \dots T_{k-1} T_{k+1} \dots T_{n+1} = 0$ as follows: Show that for every k, we have

   (a) $\texttt{range}\ (T_1 T_2 \dots T_k) \subseteq \texttt{range}\ (T_1 T_2 \dots T_{k-1})$, and

   (b) $\texttt{range}\ (T_1 T_2 \dots T_k) \subseteq \texttt{null}\ (T_{k+1} T_{k+2} \dots T_n)$.

   Then argue that for some k, we must have equality in (a), and explain why this implies the statement.

2. (5 points)
   Show that (i) is necessary for the previous conclusion by providing three operators (or matrices) $T_1, T_2, T_3 \in \mathcal{L}(\mathbb{R}^2)$ with $T_1 T_2 T_3 = 0$, but $T_1 T_2 \neq 0$, $T_1 T_3 \neq 0$, and $T_2 T_3 \neq 0$.

Solution

1. For (a), observe that $T_1 T_2 \dots T_k(v) = T_1 T_2 \dots T_{k-1}(T_k v) \in \texttt{range}\ T_1 T_2 \dots T_{k-1}$ for every v.
   For (b), observe that $(T_{k+1} T_{k+2} \dots T_n)(T_1 T_2 \dots T_k(v)) = T_1 T_2 \dots T_n(v) = 0$.
   If we have $\subsetneq$ for every k in (a), then

   $$0 = \texttt{dim range}\ T_1 \dots T_{n+1} < \texttt{dim range}\ T_1 \dots T_n < \dots < \texttt{dim range}\ T_1 \leq n.$$

   Since all dimensions are integers, this implies that $\texttt{dim range}\ T_1 \dots T_k = n + 1 - k$ for all k, and in particular that T_1 is injective and thus bijective. This implies that

   $$\texttt{range}\ T_2 \dots T_{n+1} = \texttt{range}\ T_2 \dots T_{n+1} T_1 = \{0\},$$

   so $T_2 \dots T_{n+1} = 0$.
   On the other hand, if we have "=" in (a) for some k, then

   $$\texttt{range}\ (T_1 T_2 \dots T_{k-1}) = \texttt{range}\ (T_1 T_2 \dots T_k) \subseteq \texttt{null}\ (T_{k+1} T_{k+2} \dots T_n),$$

   and $T_1 \dots T_{k-1} T_{k+1} \dots T_{n+1} = T_{k+1} \dots T_{n+1} T_1 \dots T_{k-1} = 0$.

2.

$$T_1 = T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_1T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ T_1T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ T_2T_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ T_1T_2T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 7. Let V be a finite dimensional real vector space with basis $e_1, \dots, e_n$.

1. Let A be a positive bijective matrix in V. For any $v, w \in V$ expressed as coordinate vectors according to this basis, define

$$\langle v, w \rangle := v^t A w.$$

   Show that this defines an inner product.

2. Let $\langle ., . \rangle$ be an inner product in V. Show that $a_{ij} = \langle e_i, e_j \rangle$ is a positive bijective matrix such that $\langle v, w \rangle = v^t A w$.

Solution

1. Linearity in the second argument follows from linearity of matrix multiplication. Symmetry follows from (using $A^t = A$) $w^t A v = w^t A^t v = (v^t A w)^t = v^t A w$, since the transpose of a scalar is the scalar itself.

   Since A is positive, we have $v^t A v \geq 0$ for all v. It remains to show that $v^t A v \neq 0$ for $v \neq 0$. Use an ONB $b_1, \dots, b_n$ which diagonalizes A, with eigenvalues $0 < \lambda_1 \leq \dots \lambda_n$ (0 is not an eigenvalue since A is bijective). If $(w_1, \dots, w_n)$ is the coordinate vector of v in this basis, then $v^t A v = \sum \lambda_i w_i^2 > 0$.

2. Since the inner product is symmetric, we have $a_{ij} = a_{ji}$, so A is symmetric. In the basis $e_1, \dots, e_n$, each vector e_i has ith coordinate 1, and all other coordinates 0, so $e_i^t A e_j = a_{ij} = \langle e_i, e_j \rangle$. The statement $\langle v, w \rangle = v^t A w$ follows directly from linearity of inner product and matrix multiplication. Further, $v^t A v = \langle v, v \rangle > 0$ for $v \neq 0$, so A is positive. By the same argument, 0 is not an eigenvalue of A, so A is bijective.

Problem 8. Let A be an n-by-n matrix with complex entries. Prove that A is the sum of two nonsingular matrices.

Solution To prove existence, we will construct such a sum. So given a matrix A, we will give two matrices $A^{(1)}$ and $A^{(2)}$ such that $A^{(1)}$ and $A^{(2)}$ are invertible and that $A = A^{(1)} + A^{(2)}$.

We take a singular value decomposition of A. So we have $A = USV^H$ where

- U is n-by-n and orthogonal (so that $U^H U = UU^H = I$),
- V is n-by-n and orthogonal (so that $V^H V = VV^H = I$),
- S is n-by-n and diagonal with real nonnegative entries on the diagonal, $s_1, \ldots, s_n$.

Note that such a decomposition exists for any n-by-n matrix A.

Now, we create two diagonal matrices $S^{(1)}$ and $S^{(2)}$ such that (1) $S = S^{(1)} + S^{(2)}$ and (2) $S^{(1)}$ and $S^{(2)}$ have real nonzero entries on the diagonal.

To do so, we can, for example, do as follows. For all i from 1 to n, we define $s_i^{(1)}$ and $s_i^{(2)}$, the (i,i)-th entries of the matrices $S^{(1)}$ and $S^{(2)}$ respectively as follows.

- If $s_i = 0$ then we define $s_i^{(1)} = 1$ and $s_i^{(2)} = -1$. (Then, indeed, $s_i = s_i^{(1)} + s_i^{(2)}$ and (2) $s_i^{(1)}$ and $s_i^{(2)}$ are nonzeros.)
- If $s_i \neq 0$ then we define $s_i^{(1)} = s_i/2$ and $s_i^{(2)} = s_i/2$. (Then, indeed, $s_i = s_i^{(1)} + s_i^{(2)}$ and (2) $s_i^{(1)}$ and $s_i^{(2)}$ are nonzeros.)

Then we create the matrix

$$A^{(1)} = US^{(1)}V^H \quad \text{and} \quad A^{(2)} = US^{(2)}V^H.$$

Now we claime that both matrices $A^{(1)}$ and $A^{(2)}$ are invertible and that $A = A^{(1)} + A^{(2)}$

Both matrices $A^{(1)}$ and $A^{(2)}$ are invertible since they are products of three invertible matrices, or since these matrices have nonzero singular values.

$A = A^{(1)} + A^{(2)}$ since $A^{(1)} + A^{(2)} = US^{(1)}V^H + US^{(2)}V^H = U(S^{(1)} + S^{(2)})V^H = USV^H$.