University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
February 04, 2022

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
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Part I. Work **all** of problems 1 through 4.

Problem 1. Let V be a vector space of dimension n over a field F. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a basis of V and T be an operator on V. Prove: T is invertible if and only if $T\boldsymbol{v}_1, T\boldsymbol{v}_2, \ldots, T\boldsymbol{v}_n$ is linearly independent.

Problem 2.

1. Give an orthonormal basis for $\texttt{null}\ T$, where $T \in \mathcal{L}(\mathbb{C}^4)$ (with the standard inner product) and
$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

2. Prove or disprove:
There exists an inner product $\langle .,. \rangle$ on $\mathbb{R}^2$ such that for every $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we have
$$\langle v, v \rangle = |x| + |y|.$$

Problem 3. Prove or give a counterexample to each of the following statements:

1. Let $T \in \mathcal{L}(\mathbb{R}^3)$, and $\texttt{dim}\ (\texttt{null}\ T \cap \texttt{range}\ T) \geq 1$. Then T is nilpotent.
2. Let $T \in \mathcal{L}(\mathbb{R}^4)$, and $\texttt{dim}\ (\texttt{null}\ T \cap \texttt{range}\ T) \geq 2$. Then T is nilpotent.

Problem 4. Let G, O, L, Y, N, X be 6 real numbers.

Note: the letter O is not the same as the number 0. Please make sure that you see the difference between the letters O and the numbers 0.

We consider the following 6-by-6 “Go Lynx” matrix A:

$$A = \begin{pmatrix} G & O & L & Y & N & X \\ G & -O & L & -Y & N & -X \\ G & O & L & Y & -N & -X \\ G & -O & L & -Y & -N & X \\ G & O & -L & -Y & N & X \\ G & -O & -L & Y & N & -X \end{pmatrix}.$$

Compute the determinant of A. Answer needs to be a closed form algebraic formula with variables G, O, L, Y, N, X. No matrix in final answer.

Part II. Work **two** of problems 5 through 8.

Problem 5. Let $\boldsymbol{u}$ be a unit vector in an n-dimensional inner product space V over $\mathbb{R}$. Define $T \in \mathcal{L}(V)$ as:

$$T(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{u} \rangle \boldsymbol{u}, \ \boldsymbol{x} \in V$$

Show that

1. T is an isometry.
2. If $A = \mathcal{M}(T)$ is a matrix representation of T, then $\texttt{det}\ A = -1$.
3. If $S \in \mathcal{L}(V)$ is an isometry with 1 as an eigenvalue, and if the eigenspace of 1 is of dimension $n-1$, then

   $$S(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{w} \rangle \boldsymbol{w}, \ \boldsymbol{x} \in V$$

   for some unit vector $\boldsymbol{w} \in V$.

Problem 6. Let V be an n-dimensional vector space, and let $T_1, \dots, T_{n+1} \in \mathcal{L}(V)$ such that

(i) $T_i T_j = T_j T_i$ for every $1 \le i \le j \le n+1$ (the operators commute), and

(ii) $T_1 T_2 \dots T_{n+1} = 0$.

1. (15 points)
   Show that there exists some k such that $T_1 \dots T_{k-1} T_{k+1} \dots T_{n+1} = 0$ as follows: Show that for every k, we have

   (a) $\texttt{range}\ (T_1 T_2 \dots T_k) \subseteq \texttt{range}\ (T_1 T_2 \dots T_{k-1})$, and

   (b) $\texttt{range}\ (T_1 T_2 \dots T_k) \subseteq \texttt{null}\ (T_{k+1} T_{k+2} \dots T_n)$.

   Then argue that for some k, we must have equality in (a), and explain why this implies the statement.

2. (5 points)
   Show that (i) is necessary for the previous conclusion by providing three operators (or matrices) $T_1, T_2, T_3 \in \mathcal{L}(\mathbb{R}^2)$ with $T_1 T_2 T_3 = 0$, but $T_1 T_2 \neq 0$, $T_1 T_3 \neq 0$, and $T_2 T_3 \neq 0$.

Problem 7. Let V be a finite dimensional real vector space with basis $e_1, \ldots, e_n$.

1. Let A be a positive bijective matrix in V. For any $v, w \in V$ expressed as coordinate vectors according to this basis, define

$$\langle v, w \rangle := v^t A w.$$

   Show that this defines an inner product.

2. Let $\langle ., . \rangle$ be an inner product in V. Show that $a_{ij} = \langle e_i, e_j \rangle$ is a positive bijective matrix such that $\langle v, w \rangle = v^t A w$.

Problem 8. Let A be an n-by-n matrix with complex entries. Prove that A is the sum of two nonsingular matrices.