PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
FEBRUARY 11, 2021

Student ID: ______________________________

- The examination consists of four parts associated with four skills and content areas summarized to students via email. We summarize this below.
  - **Part 1: Skills and Content Area.** Produce straightforward proofs based on general metric space definitions for point-set topology and continuous functions that may, for establishing certain steps in the proofs, utilize standard results from either undergraduate or graduate analysis (MATH 4310 or 5070, respectively). **Students solve both problems in this part.**
  - **Part 2: Skills and Content Area.** Produce proofs involving commonly studied sequence/function spaces such as ℓ^p (for $1 \leq p \leq \infty$) or $\mathcal{C}^k(X, Y)$ for some $k \in \mathbb{N}$. **Students solve both problems in this part.**
  - **Part 3: Skills and Content Area.** Identify the correct theorem to apply to prove a result by proper verification of the theorem's hypothesis. The focus is on major theorems spanning all content including some of the major results from undergraduate analysis. Such theorems include, but are not limited to, the intermediate value theorem, the mean value theorem, the Fundamental Theorem of Calculus, the Arzelà–Ascoli theorem, and the contraction mapping theorem. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
  - **Part 4: Skills and Content Area.** Prove results requiring definitions and/or theorems for differentiation/integration. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
- Make sure to justify your solutions/proofs by citing theorems that you use, provide counter-examples with explanations, follow proper proof-writing techniques, etc.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of each piece of paper**; number all pages throughout; and, just in case, write your student ID on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end.
- Ask the proctor if you have any questions.

Examination committee: Troy Butler (chair), Burt Simon, Dmitriy Ostrovskiy

Part 1

Students should complete both problems.

(1) Let (X, d) be a metric space. Suppose $(x_n), (y_n) \subset X$ both converge. Use the definition of a convergent sequence to prove that $(d(x_n, y_n)) \subset \mathbb{R}$ also converges.

Note about the proof: We enumerate the steps to the proof and provide footnotes describing their necessity.

Proof.

[Step 1:] Let $x_n \to x$ and $y_n \to y$. We prove that $d(x_n, y_n) \to d(x, y)$.[1]

[Step 2:] Let $\epsilon > 0$.[2]

[Step 3:] Since $x_n \to x$, there exists N_1 such that $n \geq N_1$ implies $d(x, x_n) < \epsilon/2$. Since $y_n \to y$, there exists N_2 such that $n \geq N_2$ implies $d(y, y_n) < \epsilon/2$. Choose $N = \max\{N_1, N_2\}$.[3]

[Step 4:] Let $n \geq N$.[4]

[Step 5:] By repeated use of the triangle inequality and the symmetric property of metrics[5],

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) = d(x_n, x) + d(x, y) + d(y_n, y),$$

which implies that

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y).$$

By reversing the roles of x_n and y_n with x and y, respectively, the same argument gives

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y).$$

It follows that

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\leq d(x_n, x) + d(y_n, y) \\ &< \epsilon. \end{aligned}$$

□

[1] This step is necessary to establish key notation regarding the limit of the sequence of real numbers $(d(x_n, y_n))$.

[2] We are asked to prove convergence using the definition of a convergent sequence. It is therefore necessary, after establishing what the limit of the sequence is in Step 1, to consider an arbitrary $\epsilon > 0$.

[3] From the definition of convergence, "for each $\epsilon > 0$ there must *exist* an N..." It is necessary to determine an N. In this case, we must use the convergence of the two sequences in question to establish how we choose an N.

[4] From the definition, it is necessary to now consider an arbitrary $n \geq N$.

[5] We need to justify why we can bound $|d(x_n, y_n) - d(x, y)|$ above by $d(x_n, x) + d(y_n, y)$ in order to exploit our choice of N in Step 3.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact. Use either the sequential or covering compactness definitions to prove the standard result that $f(K) \subset Y$ is compact.

Note about the proofs: We enumerate the steps to the proof and provide several footnotes.

Proof. (Using sequential compactness)

[Step 1:] Let $(y_n) \subset f(K)$.[6]

[Step 2(a):] For each $n \in \mathbb{N}$, since $y_n \in f(K)$ there exists an $x \in K$ such that $f(x) = y_n$. Choose such an x and denote it as x_n. This defines a sequence $(x_n) \subset K$.[7]

[Step 2(b):] Since K is compact, there exists $x_{n_k} \to x \in K$. Choose such a subsequence (x_{n_k}) and $x \in K$.

[Step 2(c):] Since f is continuous and $x \in K$, $f(x_{n_k}) \to f(x) \in f(K)$.

[Step 2(d):] Since $y_{n_k} = f(x_{n_k})$, this proves that there exists a subsequence of (y_n) that converges to a point in $f(K)$. □

Proof. (Using covering compactness)

[Step 1:] Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $f(K)$.[8]

[Step 2(a):] Since f is continuous, $f^{-1}(G_\alpha)$ is open in X for each $\alpha \in \mathcal{A}$. Let $x \in K$. This implies $f(x) \in f(K) \subset \cup_{\alpha \in \mathcal{A}} G_\alpha$, which implies $f(x) \in G_\alpha$ for some $\alpha \in \mathcal{A}$. Since $f^{-1}(\cup_{\alpha \in \mathcal{A}} G_\alpha) = \cup_{\alpha \in \mathcal{A}} f^{-1}(G_\alpha)$, this implies that $\{f^{-1}(G_\alpha)\}_{\alpha \in \mathcal{A}}$ is an open cover of K.[9]

[Step 2(b):] Since K is compact, there exists a finite subcover that we can choose and denote by $\{f^{-1}(G_n)\}_{n=1}^N$ for some finite N.

[Step 2(c):] Let $y \in f(K)$ and choose the $x \in K$ such that $f(x) = y$. Since $x \in f^{-1}(G_n)$ for some $1 \leq n \leq N$, $f(x) = y \in G_n$ for the same n. This proves that $\{G_n\}_{n=1}^N$ is a finite subcover for $f(K)$. □

[6] We must begin by considering an arbitrary sequence in $f(K)$. The next step in the proof is to establish the existence of a subsequence of this sequence that converges to a point in $f(K)$. We break this next step up into several sub-parts.

[7] This establishes a corresponding sequence to $(y_n) \subset f(K)$ in the compact K, which allows us to exploit the compactness of K.

[8] We must begin by considering an arbitrary open cover of $f(K)$. The next step in the proof is to establish the existence of a finite subcover of $f(K)$. We break this next step into several sub-parts.

[9] We utilize the continuity of f to establish a link between the open cover of $f(K)$ and the pre-images of these open sets as an open cover of the compact K.

Part 2

Students should complete both problems.

(3) Let $\mathcal{C}([a,b])$ be the space of all continuous functions $[a,b] \to \mathbb{R}$ equipped with the sup-norm metric given by

$$d_\infty(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|, \ \forall f,g \in \mathcal{C}([a,b]).$$

Prove that if $(f_n) \subset \mathcal{C}([a,b])$ is Cauchy, then the set $\{f_n\}$ is uniformly equicontinuous.

Note about the proof: This proof is based on the standard definition that a family of functions $\mathcal{F} \subset \mathcal{C}([a,b])$ is *uniformly equicontinuous* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in [a,b]$ with $|x-y| < \delta$, $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$.

Proof.

[Step 1:] Let $\epsilon > 0$.

[Step 2(a):] Since (f_n) is Cauchy, we can choose N such that for all $n, m \geq N$, $d_\infty(f_n, f_m) < \epsilon/3$.

[Step 2(b):] For each n, f_n is continuous on compact $[a,b]$, so f_n is uniformly continuous for each n by a standard result. Thus, for each n, we can choose $\delta_n > 0$ such that for all $x, y \in [a,b]$ with $|x-y| < \delta_n$, $|f_n(x) - f_n(y)| < \epsilon/3$. Choose $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.

[Step 3:] Let $x, y \in [a,b]$ such that $|x-y| < \delta$.

[Step 4(a):] Let $f \in \{f_n\}$. There are two cases: $f = f_n$ for some $n \leq N$ or $f = f_n$ for some $n > N$.

[Step 4(b)] In the case that $n \leq N$, $|x-y| < \delta \leq \delta_n$ so $|f_n(x) - f_n(y)| < \epsilon/3 < \epsilon$.

[Step 4(c)] In the case that $n > N$, by repeated use of the triangle inequality we have

$$\begin{aligned} |f_n(x) - f_n(y)| &= |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)| \\ &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \end{aligned}$$

and $|f_n(x) - f_N(x)| < \epsilon/3$ and $|f_N(y) - f_n(y)| < \epsilon/3$ by the choice of N whereas $|f_N(x) - f_N(y)| < \epsilon/3$ because $|x-y| < \delta \leq \delta_N$.

Thus, in either case, $|f_n(x) - f_n(y)| < \epsilon$. ∎

(4) Let (ℓ^2, d) denote usual metric space defined by the set of square-summable real-valued sequences, i.e.,

$$\ell^2 := \{(\xi_i) \subset \mathbb{R} \,:\, \sum_{i=1}^{\infty} \xi_i^2 < \infty\},$$

and for each $x = (\xi_i), y = (\eta_i) \in \ell^2$,

$$d(x, y) := \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2 \right)^{1/2}.$$

Let G denote the set of all real-valued sequences with a finite number of non-zero terms, i.e.,

$$G := \{(\xi_i) \subset \mathbb{R} \,:\, \exists n \in \mathbb{N}, \xi_i = 0, i \geq n\}.$$

Prove that G is dense in ℓ^2.

Note about the proof: This proof is based on the sequential characterization of a set being dense in a metric space where in this case it means that $G \subset \ell^2$ is dense if for all $x = (\xi_i) \in \ell^2$ there exists $(x_n) = ((\xi_{i,n})) \subset G$ such that $x_n \to x$.

Proof.

[Step 1:] Let $x = (\xi_i) \in \ell^2$.

[Step 2:] For each $n \in \mathbb{N}$, choose $x_n = (\xi_{i,n}) \in G$ such that

$$\xi_{i,n} = \begin{cases} \xi_i, & i \leq n, \\ 0, & i > n. \end{cases}$$

[Step 3:] We now prove that $x_n \to x$ by showing that $d(x, x_n) \to 0$. First, observe that

$$\begin{aligned} d(x, x_n) &= \left(\sum_{i=1}^{\infty} |\xi_i - \xi_{i,n}|^2 \right)^{1/2} \\ &= \left(\sum_{i=n+1}^{\infty} |\xi_i|^2 \right)^{1/2} \end{aligned}$$

By a standard result from elementary analysis, $\sum_{i=1}^{\infty} \xi_i^2 < \infty$ implies that $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} \xi_i^2 \to 0$, which finishes the proof. $\square$

Part 3

Students should choose one of the following two problems to complete.

(5) If $f : [0,1] \to [0,1]$ is continuous, then there exists $x_c \in [0,1]$ such that $f(x_c) = 1 - x_c$.

Note about the proof: The intermediate value theorem is the critical result to apply here. The proof is primarily concerned with verifying the hypothesis of this theorem, which is that the function defined by $g(x) := f(x) - (1 - x)$ is continuous and that $g(0)$ and $g(1)$ are of different signs. Of course, if $f(0) = 1$ or $f(1) = 0$, there is essentially nothing to show, so this case is also considered.

Proof. If $f(0) = 1$, then $x_c = 0$. If $f(1) = 0$, then $x_c = 1$. Otherwise, $f(0) < 1$ and $f(1) > 0$. In this case, since f is continuous, by elementary analysis results, $g(x) := f(x) - (1 - x)$ defines a continuous function on $[0,1]$ that also has the properties that $g(0) = f(0) - 1 < 0$ and $g(1) = f(1) > 0$. The intermediate value theorem then implies that there exists $x_c \in [0,1]$ such that $g(x_c) = 0$, i.e., $f(x_c) - (1 - x_c) = 0$ which is equivalent to $f(x_c) = 1 - x_c$. □

(6) Let (f_n) be a sequence of Riemann integrable functions on $[0,1]$ and (F_n) a sequence of corresponding area functions

$$F_n(x) = \int_0^x f_n(y)dy,$$

and assume that for all n, $|f_{n+1}(x)| \leq |f_n(x)|$. Prove that there is a subsequence of (F_n) converging uniformly on $[0,1]$.

Note about the proof: The Arzelà-Ascoli theorem is the critical result to apply here. The proof is primarily concerned with verifying the hypothesis of this theorem, which is that the sequence of functions (F_n) are uniformly bounded and uniformly equicontinuous.

Proof.
[Step 1: Uniform boundedness] Observe that by standard properties of the integral operator that for each $n \in \mathbb{N}$,

$$|F_n(x)| \leq \int_0^1 |f_n(y)|\; dy.$$

Since f_n is Riemann integrable on $[0,1]$ for each n, a standard result is that for each n there exists a bound $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in [0,1]$. By hypothesis, $|f_{n+1}(x)| \leq |f_n(x)|$ implies $|f_n(x)| \leq |f_1(x)| \leq M_1$. Thus,

$$|F_n(x)| \leq \int_0^1 M_1\, dy = M_1 \implies \sup_{x\in[0,1]} |F_n(x)| \leq M_1.$$

Since this bound holds for all n, we see that (F_n) is uniformly bounded.

[Step 2: Uniform equicontinuity] Observe that for each $n \in \mathbb{N}$ and $x, z \in [0,1]$ (with $z > x$), standard properties of the integral operator give

$$|F_n(x) - F_n(z)| = \left| \int_x^z f_n(y)\, dy \right| \leq \int_x^z |f_n(y)|\; dy \leq M_1 \left| x - z \right|.$$

This implies (F_n) is a uniformly Lipschitz continuous sequence of functions from which uniform equicontinuity follows.

The Arzelà-Ascoli theorem therefore applies. $\square$

Part 4

Students should choose one of the following two problems to complete.

(7) Suppose $f\colon [a,b] \to \mathbb{R}$ is differentiable and that there exists a sequence of distinct points $(x_n) \subset [a,b]$ ($x_n \neq x_m$ if $n \neq m$) such that $f(x_n) = 0$ for all n. Prove that there exists a point $c \in [a,b]$ and a subsequence of $(x_n) \subset [a,b]$, (x_{n_k}) such that $x_{n_k} \to c$ and $f(c) = f'(c) = 0$.

Note about the proof: This follows from the limit definition of pointwise differentiability. The key in the proof is to use compactness to construct the sequence of points $x_n \to c$ and then use the limit definition of pointwise differentiability to finish the result.

Proof.

[Step 1: The subsequence] By the hypothesis, we can choose $(x_n) \subset [a,b]$ such that $x_n \neq x_m$ and $f(x_n) = 0$ for all n. Since $[a,b]$ is compact, there exists $x_{n_k} \to c \in [a,b]$. Choose such a subsequence and $c \in [a,b]$.

[Step 2: $f(c) = 0$] Since f is differentiable on $[a,b]$ it is continuous on $[a,b]$ by a standard result. Thus, $f(x_{n_k}) \to f(c)$. Since $f(x_{n_k}) = 0$ for all k, this implies $f(c) = 0$.

[Step 3: $f'(c) = 0$] By definition,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

By a standard limit result, this implies that any sequence of points $(y_k) \subset [a,b]$ that converges to c has the property that $f'(c) = \lim_{k\to\infty}(f(y_k) - f(c))/(y_k - c)$. If $c \notin \{x_{n_k}\}$, then choosing $y_k = x_{n_k}$ for each k gives

$$f'(c) = \lim_{k \to \infty} \frac{f(y_k) - f(c)}{y_k - c} = \lim_{k \to \infty} \frac{0}{y_k - c} = 0.$$

If $c \in \{x_{n_k}\}$, then it can only appear at most once by the assumption of (x_n) being a sequence of unique points, so choosing (y_k) as the subsequence of (x_{n_k}) where the value of c is skipped produces the same result as above. □

(8) Let f and g be Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx > \int_a^b g(x)dx$$

Prove that there exists $c, d \in [a, b]$ with $c < d$ such that $f(x) > g(x)$ on $[c, d]$.

Note about the proof: This is most easily shown following the Darboux characterization of integrability, which is equivalent to the Riemann characterization.

Proof.

[Step 1: A function with a positive integral] Define $h(x) := f(x) - g(x)$ for all $x \in [a, b]$. Since f and g are Riemann integrable, so is h by a standard result. Moreover, by the linearity of the integral operator, it follows that $\int_a^b h(x)\,dx > 0$.

[Step 2: Darboux lower sum] The equivalence of Darboux and Riemann integration implies that there exists a partition $P = \{x_n\}_{n=0}^N$ of $[a, b]$, $a = x_0 < x_1 < \cdots < x_N = b$ such that

$$L(h, P) := \sum_{n=1}^{N} \left[\inf \{h(x) \,:\, x \in [x_{n-1}, x_n]\} (x_n - x_{n-1})\right] > \frac{1}{2} \int_a^b h(x)\,dx > 0.$$

[Step 3: The subinterval] $L(h, P) > 0$ implies at least one of the terms in the sum is strictly greater than zero, which implies that $\inf \{h(x) \,:\, x \in [x_{n-1}, x_n]\} > 0$ for some n. Choosing $c = x_{n-1}$ and $d = x_n$ for this n gives the result since the infimum of $h(x) = f(x) - g(x)$ on $[c, d]$ is strictly positive. $\square$