PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
FEBRUARY 11, 2021

Student ID: ____________________________________

- The examination consists of four parts associated with four skills and content areas summarized to students via email. We summarize this below.
  - **Part 1: Skills and Content Area.** Produce straightforward proofs based on general metric space definitions for point-set topology and continuous functions that may, for establishing certain steps in the proofs, utilize standard results from either undergraduate or graduate analysis (MATH 4310 or 5070, respectively). **Students solve both problems in this part.**
  - **Part 2: Skills and Content Area.** Produce proofs involving commonly studied sequence/function spaces such as ℓ^p (for $1 \leq p \leq \infty$) or $\mathcal{C}^k(X, Y)$ for some $k \in \mathbb{N}$. **Students solve both problems in this part.**
  - **Part 3: Skills and Content Area.** Identify the correct theorem to apply to prove a result by proper verification of the theorem's hypothesis. The focus is on major theorems spanning all content including some of the major results from undergraduate analysis. Such theorems include, but are not limited to, the intermediate value theorem, the mean value theorem, the Fundamental Theorem of Calculus, the Arzelà–Ascoli theorem, and the contraction mapping theorem. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
  - **Part 4: Skills and Content Area.** Prove results requiring definitions and/or theorems for differentiation/integration. **Students are to choose to solve only one of two problems in this part. If students do both problems, then only the first one will be graded.**
- Make sure to justify your solutions/proofs by citing theorems that you use, provide counter-examples with explanations, follow proper proof-writing techniques, etc.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Please begin solution to every problem on a new page; **write only on one side of each piece of paper**; number all pages throughout; and, just in case, write your student ID on every page.
- Do not submit scratch paper or multiple alternative solutions. If you do, we will grade the first solution to its end.
- Ask the proctor if you have any questions.

Examination committee: Troy Butler (chair), Burt Simon, Dmitriy Ostrovskiy

Part 1

Students should complete both problems.

(1) Let (X, d) be a metric space. Suppose $(x_n), (y_n) \subset X$ both converge. Use the definition of a convergent sequence to prove that $(d(x_n, y_n)) \subset \mathbb{R}$ also converges.

(2) Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact. Use either the sequential or covering compactness definitions to prove the standard result that $f(K) \subset Y$ is compact.

Part 2

Students should complete both problems.

(3) Let $\mathcal{C}([a,b])$ be the space of all continuous functions $[a,b] \to \mathbb{R}$ equipped with the sup-norm metric given by

$$d_\infty(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|, \ \forall f,g \in \mathcal{C}([a,b]).$$

Prove that if $(f_n) \subset \mathcal{C}([a,b])$ is Cauchy, then the set $\{f_n\}$ is uniformly equicontinuous.

(4) Let (ℓ^2, d) denote usual metric space defined by the set of square-summable real-valued sequences, i.e.,

$$\ell^2 := \{(\xi_i) \subset \mathbb{R} \,:\, \sum_{i=1}^{\infty} \xi_i^2 < \infty\},$$

and for each $x = (\xi_i), y = (\eta_i) \in \ell^2$,

$$d(x,y) := \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2 \right)^{1/2}.$$

Let G denote the set of all real-valued sequences with a finite number of non-zero terms, i.e.,

$$G := \{(\xi_i) \subset \mathbb{R} \,:\, \exists n \in \mathbb{N}, \xi_i = 0, i \geq n\}.$$

Prove that G is dense in ℓ^2.

Part 3

Students should choose one of the following two problems to complete.

(5) If $f : [0,1] \to [0,1]$ is continuous, then there exists $x_c \in [0,1]$ such that $f(x_c) = 1 - x_c$.

(6) Let (f_n) be a sequence of Riemann integrable functions on $[0,1]$ and (F_n) a sequence of corresponding area functions

$$F_n(x) = \int_0^x f_n(y)dy,$$

and assume that for all n, $|f_{n+1}(x)| \leq |f_n(x)|$. Prove that there is a subsequence of (F_n) converging uniformly on $[0,1]$.

Part 4

Students should choose one of the following two problems to complete.

(7) Suppose $f\colon [a,b] \to \mathbb{R}$ is differentiable and that there exists a sequence of distinct points $(x_n) \subset [a,b]$ ($x_n \neq x_m$ if $n \neq m$) such that $f(x_n) = 0$ for all n. Prove that there exists a point $c \in [a,b]$ and a subsequence $(x_{n_k}) \subset [a,b]$ such that $x_{n_k} \to c$ and $f(c) = f'(c) = 0$.

(8) Let f and g be Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx > \int_a^b g(x)dx$$

Prove that there exists $c, d \in [a, b]$ with $c < d$ such that $f(x) > g(x)$ on $[c, d]$.