University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
August 9, 2021

Name: ____

Exam Rules:

- This exam is being administered remotely using Zoom. During the exam, you must be logged in to the assigned Zoom meeting with your your camera on, and you must be visible in the camera. Your microphone can be muted, but please leave Zoom audio ON so that the proctor can speak to you if needed.
- If you have any questions during the exam, please contact the exam proctor using the zoom chat, or call the proctor at the number given below.

  Committee Member Contact Information:

Name	Phone	email	Proctoring times
Julien Langou		julien.langou@ucdenver.edu	
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Jan Mandel	xxx-xxx-xxxx	jan.mandel@ucdenver.edu	10:00am-2:00pm

- If you need to be out of view of the camera for any reason (bathroom breaks, etc.), please let the proctor know by posting a message on the zoom chat.
- If you would like to work with a paper copy of the exam, please print it as soon as you receive it and inform the proctor that you are doing so before the exam begins. If the printer is in another room, let the proctor know.
- You may read the exam as soon as you receive it, but you may not start writing (even your name) until authorized to start writing.
- This is a closed book exam. You may not use any external aids during the exam, such as:
  - communicating with anyone other than the exam proctor (through text messages or emails, for example);
  - consulting the internet, textbooks, notes, solutions of previous exams, etc;
  - using calculators or mathematical software.
- You may use a tablet PC (such as an iPad or Microsoft Surface) to write your solutions. Alternatively, you can write your solutions on paper.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). If you are writing on paper, write only on one side of the paper.
- The exam will end 4 hours after it begins. At the conclusion of the exam, please email a copy of your solutions to the exam proctor. Do not leave until the proctor acknowledges that your solutions have been successfully received.

  - Your solutions need to be in a single .pdf file with the pages in the correct order. The .pdf file needs to be of good enough quality for easy grading.
  - If you cannot create a good quality .pdf file quickly, you may instead submit an imperfect scan, or even pictures of your exam, and then take more time to prepare and submit a good quality .pdf version. We will grade the better version but use the first submission to check that nothing was added or changed between versions.
  - Do not submit your scratch work.

- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Julien Langou, Yaning Liu (Chair), Jan Mandel

Before starting the exam, please sign and date the following honor statement.

Honor statement: I attest that I will not cheat and will not attempt to cheat. I attest that I will not communicate with anyone while taking the exam, I will not look at notes, textbooks, previous solutions of exams, cheat sheets, etc. I attest that I will not go on the web to find solutions. If I in some way receive information during the exam that might help with the exam, I will let the proctor know.

Part I. Work **all** of problems 1 through 4.

Problem 1. Let A be an $m \times n$ matrix over a field $\mathbb{F}$. Prove the following statements:

(a) $\text{null}A = \{\mathbf{0}\}$ if and only if the columns of A are linearly independent.

(b) If the columns of A are linearly independent, are the rows of A necessarily linearly independent?

(c) If $m < n$, then $\text{null}A \neq \{\mathbf{0}\}$.

(d) If $A = BC$, where B is $m \times m$ and C is $m \times n$, and if B is nonsingular, then $\text{null}A = \text{null}C$.

(e) Let A^* be the conjugate transpose of A. Then $\text{null}(A^*A) = \text{null}A$.

Problem 2. Let V, W be finite dimensional real inner product spaces and let $T : V \to W$ be a linear map. Fix $\boldsymbol{w} \in W$. Show that the elements $\boldsymbol{v} \in V$ for which the Euclidean norm $||T\boldsymbol{v} - \boldsymbol{w}||$ is minimal are exactly the solutions to the equation $T^*T\boldsymbol{v} = T^*\boldsymbol{w}$.

Problem 3.

For each of the following 4 statements, give either a counterexample or a reason why it is true.

1. For every real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.
2. For every symmetric real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.
3. For every complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.
4. For every symmetric complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.

Problem 4. Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Part II. Work **two** of problems 5 through 8.

Problem 5. Consider vectors in $\mathbb{C}^n$ as columns, i.e. matrices of size n by 1. Then for two vectors $\boldsymbol{u}, \boldsymbol{v}$, the product $\boldsymbol{u}\boldsymbol{v}^*$ is an n by n matrix. It is called a "rank-one matrix" when $\boldsymbol{u}$ and $\boldsymbol{v}$ are both nonzero.

(a) Suppose that $A = A^* \in \mathbb{C}^{n,n}$. Show that A is a linear combination of rank-one matrices $\boldsymbol{u}_k\boldsymbol{u}_k^*$, $k = 1, \ldots, n$, where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ is an orthonormal basis of $\mathbb{C}^n$.

(b) Suppose that $A \in \mathbb{C}^{n,n}$. Show that A is a linear combination of rank-one matrices of the form $\boldsymbol{f}_k\boldsymbol{e}_k^*$, $k = 1, \ldots, n$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ are orthonormal bases.

Problem 6. Let A be a real 3×3 symmetric matrix, whose eigenvalues are λ_1, λ_2 and λ_3. Prove the following:

(a) If the trace of A, $\mathrm{tr}A$, is not an eigenvalue of A, then $(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3) \neq 0$

(b) If $(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) \neq 0$, then the map $T : S \to S$ is an isomorphism, where S is the space of 3×3 real skew-symmetric matrices (if $W^T = -W$, then W is called skew-symmetric) and $T(W) = AW + WA$.

Problem 7.

1. Let A be an n-by-n real matrix such that all entries not on the diagonal are positive, and the sum of the entries in each row is negative.

   Calling C_j the j-th column of A, we perform the sequence of column operations:

$$
\begin{aligned}
C_2 &\leftarrow C_2 - \frac{a_{12}}{a_{11}} C_1 \\
C_3 &\leftarrow C_3 - \frac{a_{13}}{a_{11}} C_1 \\
\vdots & \qquad \vdots \\
C_n &\leftarrow C_n - \frac{a_{1n}}{a_{11}} C_1.
\end{aligned}
$$

   So starting from A, we compute A' with this sequence of operation.

Note that $a_{11} \neq 0$, so dividing by a_{11} makes sense. Also note that this sequence of column operations introduces 0 in the first row of A for all non diagonal elements. So we have

$$A' = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ a_{31} & a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nn}^{(1)} \end{pmatrix}$$

Where

$$\begin{aligned} & & \text{second column} \\ a_{22}^{(1)} &= & a_{22} - \frac{a_{12}}{a_{11}} a_{21} \\ a_{32}^{(1)} &= & a_{32} - \frac{a_{12}}{a_{11}} a_{31} \\ \dots & & \\ a_{n2}^{(1)} &= & a_{n2} - \frac{a_{12}}{a_{11}} a_{n1} \\ & & \text{third column} \\ a_{23}^{(1)} &= & a_{23} - \frac{a_{13}}{a_{11}} a_{21} \\ a_{33}^{(1)} &= & a_{33} - \frac{a_{13}}{a_{11}} a_{31} \\ \dots & & \\ a_{n3}^{(1)} &= & a_{n3} - \frac{a_{13}}{a_{11}} a_{n1} \\ & & \text{and so on} \end{aligned}$$

In short for $2 \leq i \leq n$ and $2 \leq j \leq n$, we have

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{1j}}{a_{11}} a_{i1}.$$

Let us call $A^{(1)}$ the $(n-1)$-by-$(n-1)$ matrix

$$A^{(1)} = \begin{pmatrix} a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nn}^{(1)} \end{pmatrix}$$

Prove that $A^{(1)}$ is an $(n-1)$-by-$(n-1)$ real matrix such that all entries not on the diagonal are positive, and the sum of the entries in each row is negative.

2. Let A be an n-by-n real matrix such that all entries not on the diagonal are positive, and the sum of the entries in each row is negative. Show that the determinant of A is nonzero.

(Note: you can prove the second by assuming the first.)

Problem 8.

Let n be an integer.

We consider the inner product space, $\mathcal{P}_n$, of the polynomials of degree at most n with the inner product

$$< P, Q >= \int_0^1 P(t)Q(t)dt.$$

(Where $P(t)$ and $Q(t)$ are two polynomials of degree at most n.)

We consider $\{P_0, P_1, P_2, \ldots, P_n\}$ an orthonormal basis of $\mathcal{P}_n$, (orthonormal with respect to the aforementioned inner product), and so we note that we have

$$\int_0^1 P_i(t)P_j(t)dt = \delta_{ij}, \quad 0 \leq i, j \leq n.$$

Let H be the $(n+1) \times (n+1)$ Hilbert matrix. H is defined such that its elements are

$$h_{ij} = \int_0^1 t^i t^j dt, \quad 0 \leq i, j \leq n.$$

(Note that indexing starts at 0.)

We define the $(n+1) \times (n+1)$ matrix P made of the coefficients of the orthonormal basis $\{P_0, P_1, P_2, \ldots, P_n\}$ in the basis $\{1, t, t^2, \ldots, t^n\}$. So P is such that p_{ij} is the j-th coefficient of $P_i(t)$ so that

$$P_i(t) = \sum_{j=0}^{n} p_{ij} t^j, \quad 0 \leq i \leq n.$$

(Note that, with our notation, P is a matrix, and P_i is a polynomial, and p_{ij} is a coefficient of P_i and of P.)

Show that $H^{-1} = P^T P$.
