Department of Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
July 12, 2021 – Solutions

Problem 1. Let A be a subset of a metric space (M, d) and a a point in M (but not necessarily in A). Let S be a subset of all points $b \in M$ such that $d(a, b) \leq d(a, c)$ for every $c \in A$. Prove that S is closed.
Solution: Let s be a limit point of S and let $(s_n) \subset S$ be a sequence converging to s. For any $c \in A$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(a,s) &\leq d(a,s_n) + d(s_n,s) \quad \text{(by the triangle inequality)} \\ &\leq d(a,c) + d(s_n,s) \quad \text{(since } s_n \in S) \end{aligned}$$

Taking the limit of both sides as $n \to \infty$, we get

$$d(a,s) \leq d(a,c) + \lim_{n\to\infty} d(s_n,s) = d(a,c).$$

This inequality holds for every $c \in A$, so $s \in S$. Thus, every limit point of S is in S, so S is closed.

Problem 2. Let X and Y be subsets of a metric space (M, d), $f : X \to Y$ be a function, and (x_n) be a sequence in X. Prove or disprove the following statements:

(a) (7 pts) Let x be a limit point of X. If f is continuous and (x_n) converges to x, then the sequence $(f(x_n))$ is Cauchy.

(b) (7 pts) If f is uniformly continuous and (x_n) is Cauchy, then $(f(x_n))$ is Cauchy.

(c) (6 pts) If f is uniformly continuous and $(f(x_n))$ is Cauchy, then (x_n) is Cauchy.

Solution:

(a) False. As a counterexample, let $X = (0, 1) \subset \mathbb{R}$, $Y = \mathbb{R}$ and $f : X \to Y$ be defined by $f(x) = 1/x$. Let $x_n = 1/n, n \in \mathbb{N}$. Then $x_n \to 0$ as $n \to \infty$. However, $(f(x_n)) = (n)$, which diverges.

(b) True. Let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists $\delta > 0$ such that $d(x_n, x_m) < \delta$ implies $d(f(x_n), f(x_m)) < \epsilon$. Since (x_n) is Cauchy, there exists $\bar{n} \in \mathbb{N}$ such that

$$n, m \geq \bar{n} \implies d(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \epsilon.$$

Hence, $(f(x_n))$ is Cauchy.

(c) False. As a counterexample, let $f : (0, \infty) \to (0, 1)$ be defined by $f(x) = e^{-x}$ and let $x_n = n, n \in \mathbb{N}$. Note that $f'(x) = -e^{x} \in (0, 1)$ for $x > 0$, so f is uniformly continuous on $(0, \infty)$. Note also that $f(x_n) \to 0$ as $n \to \infty$, so $(f(x_n))$ is Cauchy; however $(x_n) = (n)$ is unbounded, so is not Cauchy.

Problem 3. Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable with bounded derivative. Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.
Solution: Since g' is bounded, there exists M such that $|g'(x)| \leq M$ for all x. Observe that f is differentiable. Choose $\epsilon < 1/M$. Then, for all x,

$$f'(x) = 1 + \epsilon g'(x) \geq 1 - \epsilon M > 0.$$

Suppose $f(x) = f(y)$. Then $0 = f(x) - f(y) = f'(z)(x - y)$ for some $z \in [x, y]$. Since $f'(z) > 0$, this implies that $x - y = 0$, so $x = y$. Thus, f is one-to-one for $\epsilon < \frac{1}{M}$.

Problem 4. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for each $n \in \mathbb{N}$ and f_n converges to f pointwise, where f is continuous. Prove or disprove that f is uniformly continuous.
Solution: f is not uniformly continuous. As a counterexample, define

$$f_n(x) = \begin{cases} x^2 & -n \le x \le n \\ n^2 & \text{otherwise.} \end{cases}$$

Observe that for any x, $\lim_{n\to x} f_n(x) = x^2$. Thus, the pointwise limit of f is defined by $f(x) = x^2$, which is continuous, but not uniformly continuous. To see this, observe that for any $\delta > 0$, we can choose $x = \frac{1}{\delta}$ and $y = x + \delta/2$. Then $|x - y| = \delta/2 < \delta$, but $|f_n(x) - f_n(y)| = |x^2 - y^2| = |x + y||x - y| > \frac{2}{\delta}\frac{\delta}{2} = 1$. However, f_n is uniformly continuous for every n. To see this, observe that f_n is continuous, so is uniformly continuous on any closed interval. In particular, f_n is uniformly continuous on $[-n-1, n+1]$. Thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in [-n-1, n+1]$ and $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$. Without loss of generality, we can choose $\delta < 1$. Then, for any $x, y \in \mathbb{R}$, if $|x-y| < \delta$, then either $x, y \in [-n-1, n+1]$, in which case $|f_n(x) - f_n(y)| < \epsilon$, or $f_n(x) = f_n(y) = n^2$, in which case $|f_n(x) - f_n(y)| = 0$. In either case, $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$, so f_n is uniformly continous.

Problem 5. Prove that every sequence in $\mathbb{R}$ has a monotone subsequence.
Solution: Let (x_n) be a given sequence and define $S := \{x_n | x_n \ge x_k \text{ for all } k > n.\}$. If S has infinitely many points, then the points in S form a monotonically decreasing subsequence.
On the other hand, if S has only finitely many points, there exists $n_1 \in \mathbb{N}$ larger than any index in S. Since $x_{n_1} \notin S$, there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Continuing in this manner, for each $k = 2, 3, \ldots$, since $x_{n_k} \notin S$, we can find $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$. By construction, the subsequence (x_{n_k}) is monotonically increasing.
Alternative Solution: Let (x_n) be a given sequence. If (x_n) is unbounded from above, then for every M there exists n such that $x_n > M$. Define $n_1 = 1$ and $n_k, k > 1$ such that $x_{n_k} > x_{n_{k-1}}$. Then the subsequence (x_{n_k}) is monotonically increasing. Similarly, if (x_n) is unbounded from below, define $n_1 = 1$ and $n_k, k > 1$ such that $x_{n_k} < x_{n_{k-1}}$, $k > 1$. Then the subsequence (x_{n_k}) is monotonically decreasing.
If (x_n) is bounded there exists a subsequence $(y_n) \subseteq (x_n)$ such that $y_n \to a$ when $n \to \infty$ for some finite a. At least one of the three sets $S_+ = \{y_n | y_n > a\}$, $S_- = \{y_n | y_n < a\}$ or $S_0 = \{y_n | y_n = a\}$ is infinite. If S_0 is infinite then the subsequence $(y_n) \cap S_0$ is constant (and therefore monotonic). If S_+ is infinite, define $(z_n) = (y_n) \cap S_+$. Take $n_1 = 1$ and define n_k such that $z_{n_k} < z_{n_{k-1}}$, for $k > 1$. Such n_k always exists because $z_n \to a$ implies that for any ϵ there is N such that $0 < z_n - a < \epsilon$ for all $n \ge N$ and one can define $\epsilon = z_{n_{k-1}} - a$. Then the subsequence (z_{n_k}) is monotonically decreasing.
Similarly, if S_- is infinite, define $(z_n) = (y_n) \cap S_-$. Take $n_1 = 1$ and define n_k such that $z_{n_k} > z_{n_{k-1}}$, for $k > 1$. Such n_k always exists because $z_n \to a$ implies that for any ϵ there is N such that $0 < a - z_n < \epsilon$ for all $n \ge N$ and one can define $\epsilon = a - z_{n_{k-1}}$. Then the subsequence (z_{n_k}) is monotonically increasing.

Problem 6. Let $f(x)$, $x \ge 0$ be nonnegative and differentiable, with $|f'(x)|$ bounded and $\int_0^\infty f(x)dx < \infty$.

(a) (13 pts) Prove that $\lim_{x\to\infty} f(x) = 0$.

(b) (7 pts) If the condition $|f'(x)|$ bounded is removed, is the result in (a) still true?

Solution:

(a) Suppose $f(x) \not\to 0$. Then $\exists \epsilon > 0$ such that $\forall x\ \exists t > x$ such that $f(t) > \epsilon$. Let $t_1 > 1$ satisfy $f(t_1) > \epsilon$, and for $n > 1$ choose $t_n > t_{n-1} + 1$ with $f(t_n) > \epsilon$. Let $\delta < \min(1, \epsilon/\gamma)$ where γ is a bound on $|f'|$. By the MVT, if $x \in (t_n - \delta/2, t_n + \delta/2)$, then for some z, $f(x) = f(t_n) + f'(z)(x - t_n) \ge f(t_n) - |f'(z)||x - t_n| \ge \epsilon - \gamma\delta/2 = \epsilon/2$. Thus,

$$\int_0^\infty f(x)dx \ge \sum_{n=1}^{\infty} \int_{t_n-\delta/2}^{t_n+\delta/2} f(x)dx \ge \sum_{n=1}^{\infty} \epsilon/2 = \infty$$

which is a contradiction.

(b) If $|f'(x)|$ is not bounded, then we cannot prove $\lim_{x\to\infty} f(x) = 0$. As a counterexample, let $f : (0, \infty) \to [0, \infty)$ be defined by

$$f(x) = \begin{cases} 1 + \cos(2^n \pi(x - n)) & \text{if } x \in (n - 2^{-n}, n + 2^{-n}), n \in N \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_0^\infty f(x)dx = \sum_{n=1}^{\infty} \int_{n-2^{-n}}^{n+2^{-n}} \cos(2^n \pi(x - n))dx \leq \sum_{n=1}^{\infty} 2^{-n+1} = 2.$$

Problem 7. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $|g'(x)| \leq M$ for all $x \in \mathbb{R}$, for some positive number M. Prove there exists exactly one $x \in [0, +\infty)$ such that $x = 1 + \cos \frac{g(x)}{2M}$.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + \cos \frac{g(x)}{2M}$. Observe that $|f'(x)| = \left| \frac{g'(x)}{2M} \sin \frac{g(x)}{2M} \right| \leq \frac{M}{2M} = 1/2$ (since $|g'(x)| \leq M$ and $|\sin z| \leq 1$ for any $z \in \mathbb{R}$). Thus, f' is a contraction on $\mathbb{R}$. Since $\mathbb{R}$ is complete, by the Banach contraction principle, f has a unique fixed point $x^* \in \mathbb{R}$ (i.e., such that $x^* = f(x^*)$). Finally, note that $f(z) \geq 0$ for all $z \in \mathbb{R}$. In particular, for $z = x^*$, we have $x^* = f(x^*) \geq 0$, so $x^* \in [0, +\infty)$.

Problem 8. Let (f_n) be a sequence of differentiable functions on $[0, 1]$, and assume that for all n, $f_n(0) = f_n'(0)$. Suppose that for all $n \in \mathbb{N}$ and all $x \in [0, 1]$, $|f_n'(x)| \leq 1$. Prove that there is a subsequence of (f_n) converging uniformly on $[0, 1]$.

Solution: Observe that for $x \in [0, 1]$, $|f_n(x)| = |f_n(0) + \int_0^x f'(x)dx| \leq |f'(0)| + \int_0^x |f'(x)|dx \leq 1 + x \leq 2$. Thus, (f_n) is a bounded sequence of functions.
Observe also that by the mean-value theorem, for any $x, y \in [0, 1]$, $x < y$, there is some $z \in (x, y)$ such that

$$|f_n(y) - f_n(x)| = |f_n'(z)(y - x)| \leq |f_n'(z)||y - x| \leq |y - x|.$$

Thus, given any $\epsilon > 0$, by choosing $\delta = \epsilon$ we have that for all $n \in \mathbb{N}$ and $x, y \in [0, 1]$, $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$. Thus, (f_n) is equicontinuous on $[0, 1]$. By the Arzelá-Ascoli Theorem, (f_n) has a uniformly convergent subsequence.
