Department of Mathematical and Statistical Sciences
Applied Analysis Preliminary Exam
July 12, 2021

Name: ____

Exam Rules:

- This is a closed book exam. You may not use external aides during the exam, such as
  - communicating with anyone other than the exam proctor;
  - consulting the internet, textbooks, solutions of previous exams, etc.
  - using calculators or mathematical software.
- You have 4 hours to complete the exam. You may write your solutions on paper, or you may use a tablet device to write your solutions. If you use a tablet device, email your solutions in a single .pdf file to burt.simon@ucdenver.edu. Do not leave until the proctor acknowledges that your solutions have been received.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Each problem is worth 20 points. the weights for each part on multi-step problems are indicated in the problem.
- Be sure to show all work that is relevant for each problem, but do not turn in scratch work.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- If you use a statement from Rudin, Pugh, or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask the proctor.
- If you are writing on paper, begin each solution on a new page and write on only one side of the paper. Write legibly using a dark pencil or pen.
- If you are using a tablet computer to write your solutions, make sure each problem is separate and easy to find.
- <u>Instructions for Remote Students:</u> The following instructions apply if you are taking the exam remotely:
  - You must be logged in to the assigned Zoom meeting with your camera on, and you must be visible in the camera. Your microphone can be muted, but please leave Zoom audio ON so that the proctor can speak to you if needed.
  - If you have questions during the exam, please contact the exam proctor using the zoom chat.

Scoring

1	2	3	4	5	6	7	8	$\sum$

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Part I. Work **all** of problems 1 through 4.

Problem 1. Let A be a subset of a metric space (M, d) and a a point in M (but not necessarily in A). Let S be a subset of all points $b \in A$ such that $d(a, b) \leq d(a, c)$ for every $c \in A$. Prove that S is closed.

Problem 2. Let X and Y be subsets of a metric space (M, d), $f : X \to Y$ be a function, and (x_n) be a sequence in X. Prove or disprove the following statements:

(a) (7 pts) Let x be a limit point of X. If f is continuous and (x_n) converges to x, then the sequence $(f(x_n))$ is Cauchy.

(b) (7 pts) If f is uniformly continuous and (x_n) is Cauchy, then $(f(x_n))$ is Cauchy.

(c) (6 pts) If f is uniformly continuous and $(f(x_n))$ is Cauchy, then (x_n) is Cauchy.

Problem 3. Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable with bounded derivative. Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.

Problem 4. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for each $n \in \mathbb{N}$ and f_n converges to f pointwise, where f is continuous. Prove or disprove that f is uniformly continuous.

Part II. Work **two** of problems 5 through 8.

Problem 5. Prove that every sequence in $\mathbb{R}$ has a monotone subsequence.

Problem 6. Let $f(x),\ x \geq 0$ be nonnegative and differentiable, with $|f'(x)|$ bounded and $\int_0^\infty f(x)dx < \infty$.

(a) (13 pts) Prove that $\lim_{x\to\infty} f(x) = 0$.

(b) (7 pts) If the condition $|f'(x)|$ bounded is removed, is the result in (a) still true?

Problem 7. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $|g'(x)| \leq M$ for all $x \in \mathbb{R}$, for some positive number M. Prove there exists exactly one $x \in [0, +\infty)$ such that $x = 1 + \cos \frac{g(x)}{2M}$.

Problem 8. Let (f_n) be a sequence of differentiable functions on $[0, 1]$, and assume that for all n, $f_n(0) = f_n'(0)$. Suppose that for all $n \in \mathbb{N}$ and all $x \in [0, 1]$, $|f_n'(x)| \leq 1$. Prove that there is a subsequence of (f_n) converging uniformly on $[0, 1]$.
