University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 29, 2021

Name: ____

Exam Rules:

- This exam is being administered remotely using Zoom. During the exam, you must be logged in to the assigned Zoom meeting with your your camera on, and you must be visible in the camera. Your microphone can be muted, but please leave Zoom audio ON so that the proctor can speak to you if needed.
- If you have any questions during the exam, please contact the exam proctor using the zoom chat, or call the proctor at the number given below.

  Committee Member Contact Information:

Name	Phone	email	Proctoring times
Julien Langou	xxx-xxx-xxxx	julien.langou@ucdenver.edu	10:00am-12:00pm
Yaning Liu	xxx-xxx-xxxx	yaning.liu@ucdenver.edu	12:00pm-2:00pm
Jan Mandel		jan.mandel@ucdenver.edu	

- If you need to be out of view of the camera for any reason (bathroom breaks, etc.), please let the proctor know by posting a message on the zoom chat.
- If you would like to work with a paper copy of the exam, please print it as soon as you receive it and inform the proctor that you are doing so before the exam begins. If the printer is in another room, let the proctor know.
- You may read the exam as soon as you receive it, but you may not start writing (even your name) until authorized to start writing.
- This is a closed book exam. You may not use any external aids during the exam, such as:
  - communicating with anyone other than the exam proctor (through text messages or emails, for example);
  - consulting the internet, textbooks, notes, solutions of previous exams, etc;
  - using calculators or mathematical software.
- You may use a tablet PC (such as an iPad or Microsoft Surface) to write your solutions. Alternatively, you can write your solutions on paper.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). If you are writing on paper, write only on one side of the paper.
- The exam will end 4 hours after it begins. At the conclusion of the exam, please email a copy of your solutions to the exam proctor. Do not leave until the proctor acknowledges that your solutions have been successfully received.

- Your solutions need to be in a single .pdf file with the pages in the correct order. The .pdf file needs to be of good enough quality for easy grading.
- If you cannot create a good quality .pdf file quickly, you may instead submit an imperfect scan, or even pictures of your exam, and then take more time to prepare and submit a good quality .pdf version. We will grade the better version but use the first submission to check that nothing was added or changed between versions.
- Do not submit your scratch work.

- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Julien Langou, Yaning Liu (Chair), Jan Mandel

Before starting the exam, please sign and date the following honor statement.

Honor statement: I attest that I will not cheat and will not attempt to cheat. I attest that I will not communicate with anyone while taking the exam, I will not look at notes, textbooks, previous solutions of exams, cheat sheets, etc. I attest that I will not go on the web to find solutions. If I in some way receive information during the exam that might help with the exam, I will let the proctor know.

Part I. Work **all** of problems 1 through 4.

Problem 1. Suppose $v_1, \ldots, v_m$ is linearly independent in V, $w \in V$, and $v_1 + w, \ldots, v_m + w$ is linearly dependent. Prove that $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Problem 2. Suppose that V is inner product space, real or complex.

1. (10 points) Prove that if $u, v, w \in V$, then

$$\left\| w - \frac{1}{2}(u+v) \right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

2. (10 points) Suppose that S is a subset of V such that if $u, v \in S$ then $\frac{1}{2}(u+v) \in S$, and $w \in V$. Show that there is at most one point u in S that is closest to w, that is, such that

$$\|w-u\| \leq \|w-y\| \text{ for all } y \in S.$$

Problem 3. Let A be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix Q and a unique positive definite symmetric matrix B such that $A = QB$.

Problem 4. We are in 2021, so let $A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$. Define

$$\begin{array}{rcl} T : \mathcal{M}_2(\mathbb{R}) & \to & \mathcal{M}_2(\mathbb{R}) \\ B & \mapsto & AB - BA \end{array}$$

1. (5 points) Fix an ordered basis $\mathcal{B}$ of $\mathcal{M}_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.
2. (5 points) Give the eigenvalues of T.
3. (5 points) Compute a basis for each of the eigenspaces of T.
4. (5 points) Give the minimal and characteristic polynomials of T and the Jordan form for T. Say whether T is diagonalizable or not.

Part II. Work **two** of problems 5 through 8.

Problem 5. Define $\mathbb{R}^{n\times n}$ to be the space of all real n-by-n matrices, suppose $S \in \mathbb{R}^{n\times n}$, and define the linear mapping

$$\mathcal{T} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad \mathcal{T} : P \mapsto PS + SP$$

1. (10 points) Prove that if λ is eigenvalue of S, u is the corresponding eigenvector, and $u \in \operatorname{null}(\mathcal{T}P)$, then Pu is also an eigenvector of S, with eigenvalue $-\lambda$.
2. (10 points) Prove that if S is symmetric positive definite, then the mapping $\mathcal{T}$ is injective.

Problem 6.

Let V be a vector space of dimension n over a field F. For any nilpotent operator T on V, define the smallest integer p such that $T^p = 0$ as the index of nilpotency of T.

1. (6 points) Suppose that N is nilpotent of index p. If $\boldsymbol{v} \in V$ is such that $N^{p-1}(\boldsymbol{v}) \neq 0$, prove that
$$\{\boldsymbol{v}, N(\boldsymbol{v}), \ldots, N^{p-1}(\boldsymbol{v})\}$$
is linearly independent.
2. (7 points) Show that N is nilpotent of index n if and only if there is an ordered basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of V such that the matrix of N with respect to the basis is of the form
$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$
3. (7 points) Show that an $n \times n$ matrix M over F is such that $M^n = 0$ and $M^{n-1} \neq 0$ if and only if M is similar to a matrix of the above form.

Problem 7. Let S, T be two normal operators in the complex finite dimensional inner product space V such that $ST = TS$. Prove that there is a basis for V consisting of vectors that are eigenvectors of both S and T.

Problem 8. Let $\mathcal{M}_2(\mathbb{C})$ be the set of 2-by-2 matrices with coefficients in $\mathbb{C}$, and $A \in \mathcal{M}_2(\mathbb{C})$. Define

$$\mathcal{S} = \left\{ N \in \mathcal{M}_2(\mathbb{C}) | B = \begin{pmatrix} A & N \\ 0 & A \end{pmatrix} \text{ is diagonalizable} \right\}.$$

1. (10 points) Suppose that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1 \neq \lambda_2,$$

   and prove that $\mathcal{S}$ is a 2-dimensional subspace of $\mathcal{M}_2(\mathbb{C})$. Hint: First consider $B \in \mathcal{S}$ and find the conditions which N must satisfy for the eigenspaces of B to have the required dimensions such that B is diagonalizable.

2. (10 points) Prove that for any $A \in \mathcal{M}_2(\mathbb{C})$ with two distinct eigenvalues λ_1 and λ_2, $\mathcal{S}$ is a 2-dimensional subspace of $\mathcal{M}_2(\mathbb{C})$. Hint: Transform this to the previous case.
